Oscillating Sphere

1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model an oscillating sphere acoustic problem using an Indirect model.
- Derive analytical solutions for the interior (bounded) and exterior (unbounded) problems of the oscillating sphere.
- Validate Coustyx program by comparing the results from Coustyx to the analytical solutions.

2 Model description

We model a sphere of radius $a = 1\,\text{m}$. The fluid medium surrounding the sphere is air with sound speed $c = 343\,\text{m/s}$ and mean density $\rho_o = 1.21\,\text{kg/m}^3$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\,Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c$. The sphere is oscillating with unit velocity in the z direction, that is $v_z = 1\,\text{m/s}$. The BE mesh of the sphere is shown in Figure 1.

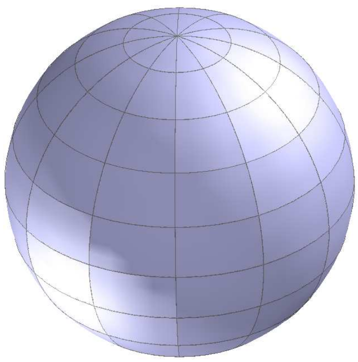

Figure 1: Boundary element mesh for a sphere of unit radius.

The BE mesh has quadratic coordinate connectivity as well as quadratic variable node connectivity. Coustyx indirect BE method is used to solve the exterior and interior problems simultaneously.

3 Boundary Conditions

The boundary condition applied on the sphere is an "Uniform Velocity of Continuous type" with velocity component only in z direction, that is, $v_z^+ = v_z^- = v_z$, where $+$ and $-$ correspond to the sides of the mesh in the same and opposite directions of the mesh normal.

4 Analytical solution

The oscillating sphere problem has exact solutions.

For the exterior problem, the pressure at point (x,y,z) at a distance r from the center of the sphere is given by

$$\tilde{p}_{ext}(x,y,z) = 3v_z(ikZ_o)z\frac{h_1(kr)}{kr(h_0(ka)-h_2(ka))}$$

where h_l is the spherical Hankel function of order l. This expression can be reduced to

$$\tilde{p}_{ext}(x,y,z) = \frac{a^2}{r^3}v_z z(ikaZ_o)e^{ik(r-a)}(1-ikr)\frac{(k^2a^2-2-2ika)}{(k^4a^4+4)} \quad (1)$$

The velocity field corresponding to the exterior pressure field is

$$\vec{v}_{ext} = \tilde{v}_x\hat{e}_1 + \tilde{v}_y\hat{e}_2 + \tilde{v}_z\hat{e}_3$$

where the velocity field components are given by

$$\tilde{v}_x(x,y,z) = v_z\frac{zx}{r^3}\left[\frac{r(h_0(kr)-2h_2(kr))-3h_1(kr)}{h_0(ka)-2h_2(kr)}\right]$$

or,

$$\tilde{v}_x(x,y,z) = v_z(k^2r^2-3+3ikr)zxe^{ik(r-a)}a^3\frac{(k^2a^2-2-2ika)}{r^5(k^4a^4+4)} \quad (2)$$

$$\tilde{v}_y(x,y,z) = v_z\frac{zy}{r^3}\left[\frac{r(h_0(kr)-2h_2(kr))-3h_1(kr)}{h_0(ka)-2h_2(kr)}\right]$$

or,

$$\tilde{v}_y(x,y,z) = v_z(k^2r^2-3+3ikr)zye^{ik(r-a)}a^3\frac{(k^2a^2-2-2ika)}{r^5(k^4a^4+4)} \quad (3)$$

$$\tilde{v}_z(x,y,z) = \frac{v_z}{h_0(ka)-2h_2(kr)}\left[\frac{z^2}{r^2}(h_0(kr)-2h_2(kr)) + \frac{3(x^2+y^2)}{r^3}h_1(kr)\right]$$

or,

$$\tilde{v}_z(x,y,z) = v_z(k^2r^2-3+3ikr)z^2e^{ik(r-a)}a^3\frac{(k^2a^2-2-2ika)}{r^5(k^4a^4+4)} + (1-ikr)e^{ik(r-a)}a^3\frac{(k^2a^2-2-2ika)}{r^3(k^4a^4+4)} \quad (4)$$

For the interior problem, the pressure at point (x,y,z) at a distance r from the center of the sphere is given by

$$\tilde{p}_{int}(x,y,z) = 3v_z z(ikZ_o)\frac{j_1(kr)}{kr(j_0(ka)-2j_2(ka))} \quad (5)$$

where j_l is the spherical Bessel function of order l.
The velocity field corresponding to the interior pressure field is

$$\vec{v}_{int} = \tilde{v}_x\hat{e}_1 + \tilde{v}_y\hat{e}_2 + \tilde{v}_z\hat{e}_3$$

where the velocity field components are given by

$$\tilde{v}_x(x,y,z) = v_z\frac{zx}{r^3}\left[\frac{r(j_0(kr)-2j_2(kr))-3j_1(kr)}{j_0(ka)-2j_2(ka)}\right] \quad (6)$$

$$\tilde{v}_y(x,y,z) = v_z\frac{zy}{r^3}\left[\frac{r(j_0(kr)-2j_2(kr))-3j_1(kr)}{j_0(ka)-2j_2(ka)}\right] \quad (7)$$

$$\tilde{v}_z(x,y,z) = \frac{v_z}{j_0(ka)-2j_2(ka)}\left[\frac{z^2}{r^2}(j_0(kr)-2j_2(kr)) + \frac{3(x^2+y^2)}{r^3}j_1(kr)\right] \quad (8)$$

The normal pressure gradient at any point in the space (interior and exterior domains) is

$$\tilde{p}_n = ikZ_o(\vec{v}.\hat{n})$$

where $\vec{v} = \vec{v}_{ext}$ or $\vec{v}_{int}$, $\hat{n}$ is the unit normal at the point.

For the current model since the mesh normal is opposite to the radial direction, the single-layer potential σ on the surface $r = a$ is given by

$$\sigma = -((\tilde{p}_n)_{ext}(r = a) - (\tilde{p}_n)_{int}(r = a)) \tag{9}$$

The double-layer potential μ on the surface $r = a$ is

$$\mu = -(\tilde{p}_{ext}(r = a) - \tilde{p}_{int}(r = a)) \tag{10}$$

The analytical expression for the radiated power by an oscillating sphere is given as,

$$W = \frac{2\pi}{3}\rho_o c v_z^2 \frac{k^4 a^6}{k^4 a^4 + 4} \tag{11}$$

5 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx Indirect model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 54.59Hz$ using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".

Coustyx Indirect model uses the Indirect BE method to solve for the surface potentials μ and σ. These are, in turn, used to compute field pressures at the specified points. The indirect BE solution on the sphere surface at point (0, 0, 1) μ = (-82.21, 768.05) compares well with the analytical solution (-83.01, 771.71).

Figure 2 shows field point pressure variation with distance from both Coustyx and Analytical methods. Since Coustyx Indirect method solves both the interior and exterior problems simultaneously, we can compute field point pressures anywhere in the space from the surface potentials μ and σ. For $r/a < 1$, the analytical solution from the interior problem of the oscillating sphere is compared with the Coustyx solution. For $r/a \geq 1$, exterior problem analytical solution is used. The comparisons show very good agreement between the solutions computed from Coustyx and analytical expressions. The radiated power computed by Coustyx, 169.94, matches well with the exact analytical solution for exterior problem, 173.85.

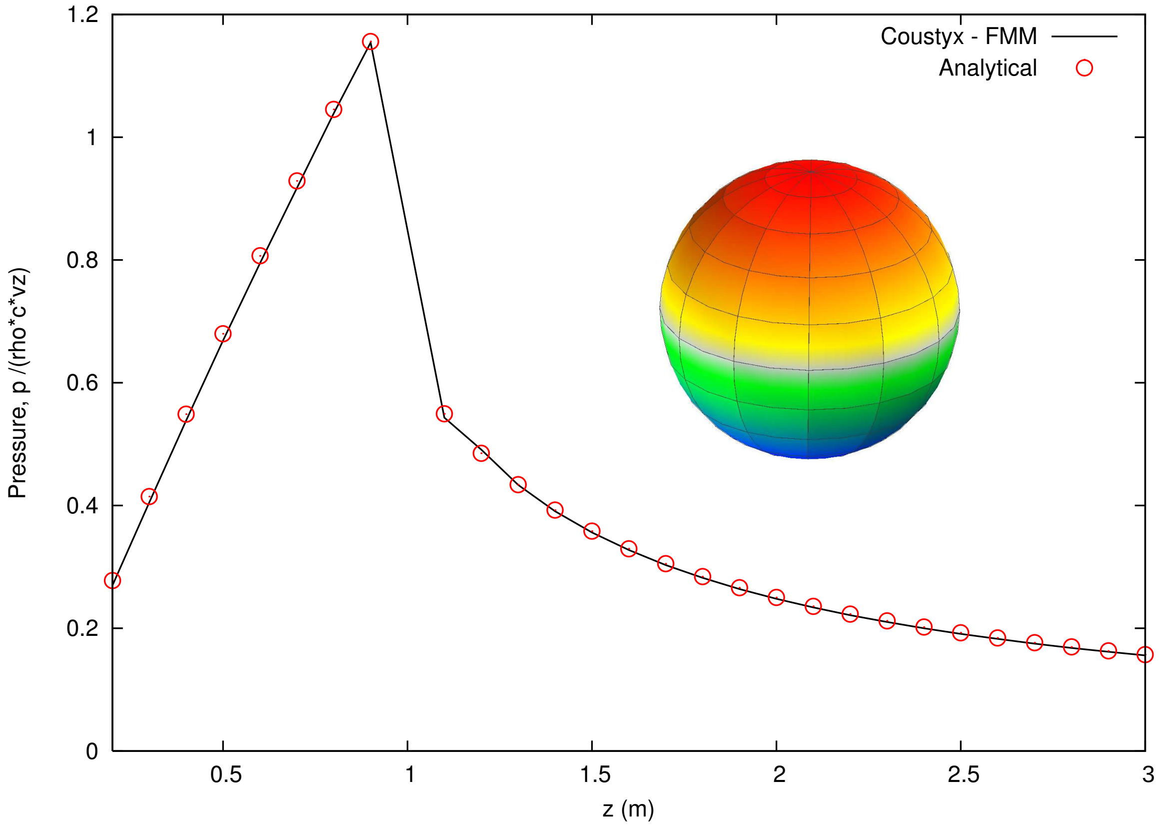

Figure 2: Field pressure comparisons at points (0,0,z) from Coustyx and analytical methods.