Pulsating Sphere

1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model a pulsating sphere acoustic problem using an Indirect model.
- Derive analytical solutions for the interior (bounded) and exterior (unbounded) problems of the pulsating sphere.
- Validate Coustyx software by comparing Coustyx results to analytical solutions.

2 Model description

We model a sphere of radius $a = 1\,\mathrm{m}$. The fluid medium surrounding the sphere is air with sound speed $c = 343\,\mathrm{m/s}$ and mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\,Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c$. The sphere is pulsating with a radial velocity $v_r = -1\,\mathrm{m/s}$. The BE mesh of the sphere is shown in Figure 1.

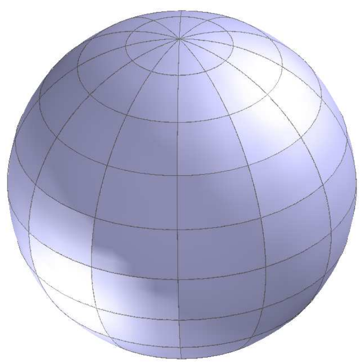

Figure 1: Boundary element mesh for a sphere of unit radius.

The BE mesh has quadratic coordinate connectivity as well as quadratic variable node connectivity. Coustyx indirect BE method is used to solve the exterior and interior problems simultaneously.

3 Boundary Conditions

The boundary condition applied on the sphere is an “Uniform Normal Velocity of Continuous type”, that is, $v_n^+ = v_n^- = v_n$, where $v_n = -v_r$ is the normal velocity on the surface of the sphere, $+$ and $-$ correspond to the sides of the mesh in the same and opposite directions of the mesh normal.

4 Analytical solution

The pulsating sphere with a uniform radial velocity has exact solutions. For the exterior problem, the pressure at a distance r from the center of the sphere is given by

$$\tilde{p}_{ext}(r) = \frac{a}{r} v_r Z_o \frac{-ika}{1 - ika} \exp\left(ik(r - a)\right) \tag{1}$$

$$\left(\frac{\partial \tilde{p}}{\partial r}\right)_{ext} = \frac{a}{r} v_r Z_o \frac{-ika}{1-ika} \exp\left(ik(r-a)\right)(ik - \frac{1}{r}) \tag{2}$$

For the interior problem, the pressure at a distance r from the center of the sphere is given by

$$\tilde{p}_{int}(r) = iZ_o v_r \left(\frac{a}{r}\right) \frac{ka\sin(kr)}{ka\cos ka - \sin ka} \tag{3}$$

$$\left(\frac{\partial \tilde{p}}{\partial r}\right)_{int} = iZ_o v_r \left(\frac{a}{r}\right)^2 \frac{kr\cos(kr) - \sin(kr)}{ka\cos ka - \sin ka} \tag{4}$$

The normal pressure gradient on the surface of the sphere is given by,

$$\tilde{p}_n(r) = \frac{\partial p}{\partial n} = \pm\frac{\partial p}{\partial r} \tag{5}$$

where it is $+$ when the mesh normal is in the radial direction, and $-$ when mesh normal is in the opposite direction.
For the current model since the mesh normal is opposite to the radial direction, the single-layer potential σ on the surface is given by

$$\sigma = -((\tilde{p}_n)_{ext}(r=a) - (\tilde{p}_n)_{int}(r=a)) \tag{6}$$

The double-layer potential μ on the surface is

$$\mu = -(\tilde{p}_{ext}(r=a) - \tilde{p}_{int}(r=a)) \tag{7}$$

The analytical expression for the radiated power by a pulsating sphere is given as,

$$W = 2\pi\rho_o c v_r^2 \frac{k^2a^4}{1+k^2a^2} \tag{8}$$

5 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx Indirect model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 54.59Hz$ using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".
Coustyx uses the Indirect BE method to solve for the surface potentials μ and σ. These are, in turn, used to compute field pressures at the specified points. The Indirect BE solution on the sphere surface at point (0, 1, 0) μ = (205.15, 953.77) compares well with the analytical solution (207.515, 952.087).
Figure 2 shows field point pressure variation with distance from both Coustyx and Analytical methods. Since Coustyx Indirect method solves both the interior and exterior problems simultaneously, we can compute field point pressures anywhere in the space from the surface potentials μ and σ. For $r/a < 1$, the analytical solution from the interior problem of pulsating sphere is compared with the Coustyx solution. For $r/a \geq 1$ exterior problem analytical solution is used. The comparisons show very good agreement between the solutions computed from Coustyx and analytical expressions. The radiated power computed by Coustyx is 1284.99 Watts, which matches well with the exact analytical solution of 1303.86 Watts for an exterior problem.

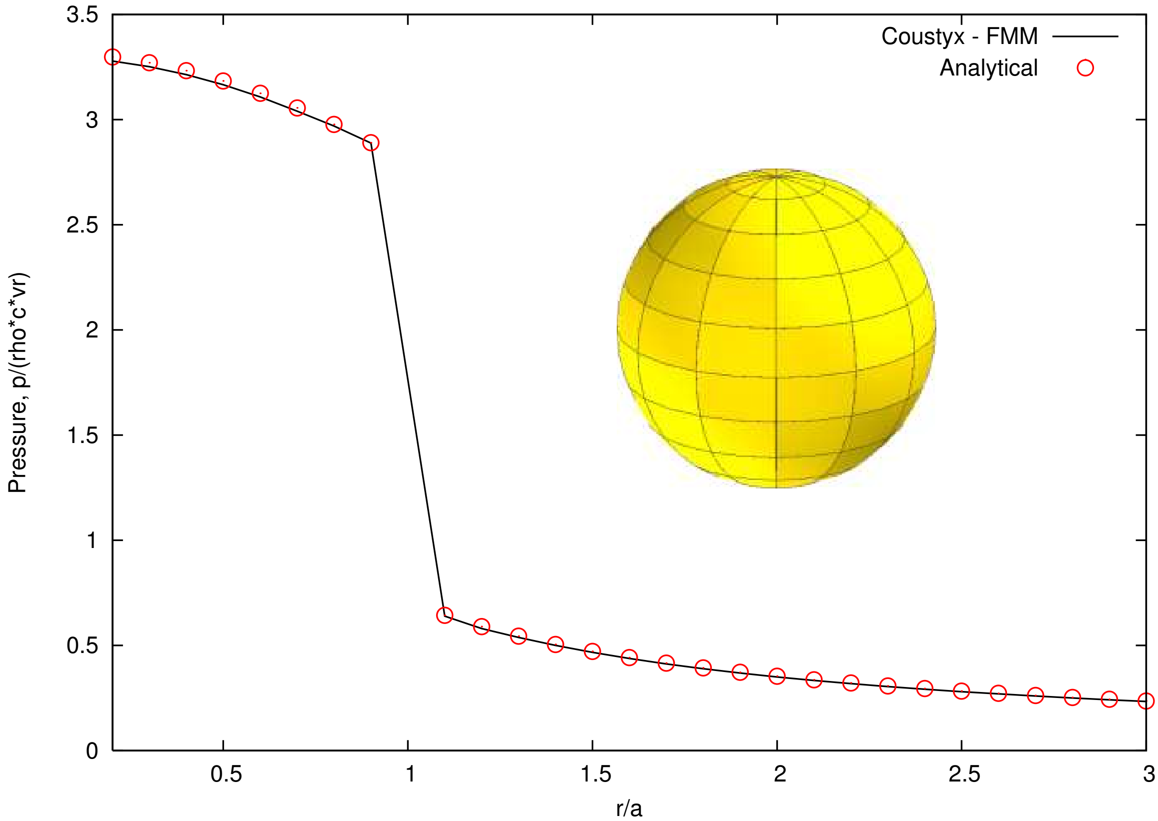

Figure 2: Field point pressure comparisons with distance, r (from center of sphere), from Coustyx and Analytical methods.