Radiation from a Sphere with Spherical Harmonic Excitation

1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model a sphere with spherical harmonic excitation using Indirect model and solve for the radiated power.
- Demonstrate the ability of Coustyx to define complex boundary conditions using *scripts*.
- Demonstrate the ability to define discontinuous boundary conditions, which can be used effectively to suppress large errors due to non-uniqueness issue at cavity resonances.
- Validate Coustyx software by comparing the results from Coustyx to the analytical solutions.

2 Model description

We model a sphere of radius $a = 1\,\mathrm{m}$. The fluid medium surrounding the sphere is air with sound speed $c = 343\,\mathrm{m/s}$ and mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\,Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c$. The BE mesh of the sphere is shown in Figure 1.

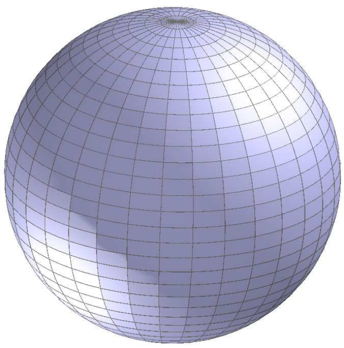

Figure 1: Boundary element mesh of a sphere with unit radius.

A radial velocity distribution, $\tilde{v}_r$, represented by arbitrary spherical harmonics is applied on the sphere,

$$\tilde{v}_r(\theta, \phi) = -u_0 P_l^m(\cos\theta)\cos(m\phi) \tag{1}$$

where P_l^m is the associated Legendre function of degree $l = 4$ and order $m = 2$, and $u_0 = 1$ is a scalar coefficient.

3 Boundary Conditions/Non-uniqueness Issue

Coustyx Indirect BE method is used to solve this acoustic radiation problem. The indirect BE method solves both the exterior and interior problems simultaneously. Near the resonance frequencies of the cavity, the exterior domain solution might contain large errors due to the non-uniqueness issue. Discontinuous boundary conditions are applied to separate the interior resonance effects on the exterior solution. These boundary conditions (BCs) have different values on each side of the

boundary. The exterior surface of the sphere is applied with spherical harmonic excitation and the interior surface with a zero velocity excitation.
The boundary condition applied on the sphere in Coustyx is of "Discontinuous" type. The side boundary conditions are

side 1 : Uniform Normal Velocity, $v_n^+ = 0$.

side 2 : Non-Uniform Normal Velocity, spherical harmonic excitation described by a script, $v_n^- = -\tilde{v}_r$. Scripts can used to define complex boundary conditions in Coustyx.

Note that *side 1* of the boundary is on the positive side (+) of the normal and *side 2* is on the negative side (−) (refer Figure 2). Since the mesh normal is in the negative radial direction, *side 1* is on the interior side of the sphere and *side 2* is on the exterior side.

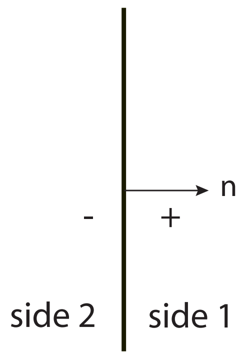

Figure 2: Description of indirect BE discontinuous side boundary conditions.

4 Analytical solution

The exact solution to the Helmholtz equation in the exterior domain can be assumed to be of the form

$$p(r,\theta,\phi) = A_l^m u_0 P_l^m(\cos\theta)\cos(m\phi)h_l^1(kr) \tag{2}$$

where $h_l^1(kr)$ is the spherical Hankel function of the first kind of order l and A_l^m is a constant dependent on (l,m). We need to solve for A_l^m to get analytical expression for pressure in the exterior domain.
The pressure gradient in the radial direction on the surface of the sphere is

$$\frac{\partial p}{\partial r}(r,\theta,\phi) = A_l^m u_0 P_l^m(\cos\theta)\cos(m\phi)\left[\frac{kl(h_{l-1}^1(kr) - h_{l+1}^1(kr)) - kh_{l+1}^1(kr)}{(2l+1)}\right] \tag{3}$$

The specified radial velocity ($\tilde{v}_r$) and the pressure gradient in the radial direction on the surface of the sphere ($r = a$) are related and can be used to obtain A_l^m, that is,

$$\tilde{v}_r(\theta,\phi)(ikZ_0) = \frac{\partial p}{\partial r}(a,\theta,\phi) \tag{4}$$

$$A_l^m = \frac{-(ikZ_0)(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)} \tag{5}$$

Thus, the analytical expression for pressure at any point (r,θ,ϕ) in the exterior domain is given by

$$p(r,\theta,\phi) = \left[\frac{-(ikZ_0)(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right] u_0 P_l^m(\cos\theta)\cos(m\phi)h_l^1(kr) \tag{6}$$

The velocity at the exterior point (r,θ,ϕ) is

$$\overrightarrow{v}(r,\theta,\phi) = 1/(ikZ_0)\overrightarrow{\nabla}p(r,\theta,\phi) \tag{7}$$

$$v_r(r,\theta,\phi) = -u_0 \left[\frac{klh_{l-1}(kr) - klh_{l+1}(kr) - kh_{l+1}(kr)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right] P_l^m(\cos\theta)\cos(m\phi) \tag{8}$$

$$v_\theta(r,\theta,\phi) = (1/r)u_0 \left[\frac{h_l(kr)(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right] \left[\frac{lP_{l-1}(\cos\theta) - l\cos\theta P_l^m(\cos\theta)}{\sin\theta}\right] \cos(m\phi) \tag{9}$$

$$v_\phi(r,\theta,\phi) = (m/r)u_0 \left[\frac{h_l(kr)(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right] \frac{P_l^m(\cos\theta)}{\sin\theta}\sin(m\phi) \tag{10}$$

where v_r, v_θ and v_ϕ are the components of velocity in the spherical coordinate system.
The pressure and velocity in Cartesian coordinates can be obtained by applying the following transformations:

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arccos(z/\sqrt{x^2+y^2+z^2}), 0 \le \theta \le \pi \\ \phi &= \arctan(y/x), 0 \le \phi \le 2\pi \end{aligned} \tag{11}$$

$$p(x,y,z) = p(r(x,y,z), \theta(x,y,z), \phi(x,y,z)) \tag{12}$$

$$v_x(x,y,z) = \frac{x}{r}v_r + \frac{xz}{r\sqrt{x^2+y^2}}v_\theta - \frac{y}{\sqrt{x^2+y^2}}v_\phi \tag{13}$$

$$v_y(x,y,z) = \frac{y}{r}v_r + \frac{yz}{r\sqrt{x^2+y^2}}v_\theta + \frac{x}{\sqrt{x^2+y^2}}v_\phi \tag{14}$$

$$v_z(x,y,z) = \frac{z}{r}v_r - \frac{\sqrt{x^2+y^2}}{r}v_\theta \tag{15}$$

The pressure on the exterior surface of the sphere can be written in terms of the spherical harmonic excitation on the exterior surface as

$$p(a,\theta,\varphi) = f.(ikz_o)v_r(\theta,\varphi) = (f_r + if_i).(ikz_o)v_r(\theta,\varphi) = (kz_o)(-f_i + f_r)v_r(\theta,\varphi)$$

where the factor $f = (f_r + if_i)$ is

$$f = \frac{(2l+1)h_l^1(ka)}{\left[kl(h_{l-1}^1(ka) - h_{l+1}^1(ka)) - kh_{l+1}^1(ka)\right]}$$

Using the orthogonality of the associated Legendre functions, that is,

$$\int_{-1}^{1} [P_l^m(x)]^2 dx = 1$$

the integral over the norm of the radial velocity is reduced to

$$\begin{aligned} \int_S |v_r|^2 dS &= \int\int u_0^2 [P_l^m(\cos\theta)]^2 [\cos^2 m\phi] a^2 \sin\theta d\theta d\varphi \\ &= C_m \pi a^2 u_0^2 \end{aligned}$$

where $C_m = 1$ for $m \neq 0$, and $C_m = 2$ for $m = 0$.
The analytical expression for the radiated power (W) due to spherical harmonic excitation on a sphere is derived to be,

$$\begin{aligned} W &= \tfrac{1}{2}\mathrm{Re}\{\int_S pv^* dS\} \\ &= -\tfrac{1}{2}kz_o f_i \int_S |v_r|^2 dS \\ &= -C_m \tfrac{1}{2}kz_o f_i \pi a^2 u_0^2 \end{aligned} \tag{16}$$

where $C_m = 1$ for $m \neq 0$, and $C_m = 2$ for $m = 0$.
The radiation efficiency σ_0 is given by,

$$\sigma_0 = \frac{W}{\frac{1}{2}\int_S z_o v_r^2(\theta,\varphi) dS} = -kf_i \tag{17}$$

5 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx Indirect model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In the demo model, the analysis is performed for the frequency range $ka = 0.2$ to $ka = 10$ with a resolution $\Delta = 0.2$ using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".

The indirect BE model solves for the surface potentials μ and σ. These are, in turn, used to compute radiated sound power at analysis frequencies. Please note that Coustyx doesn't compute input sound power for discontinuous velocity BC, as the input sound power definition is arbitrary when both the interior and exterior surfaces of the sphere have different velocities. So the default zero output values for input sound power and radiation efficiency should be ignored.

Figure 3 shows comparisons of radiated sound power computed from both Coustyx and analytical methods for a frequency range $ka = 0.2$ to $ka = 10$. The comparisons show very good agreement between the methods. From the Figure 3 one can see that the interior cavity resonances at $ka = 5.6$ and $ka = 9.8$ are suppressed in the solution from Coustyx due to the use of discontinuous BCs. One can verify the existence of these resonances by changing the boundary condition from discontinuous BC to continuous normal velocity BC with spherical harmonic excitation.

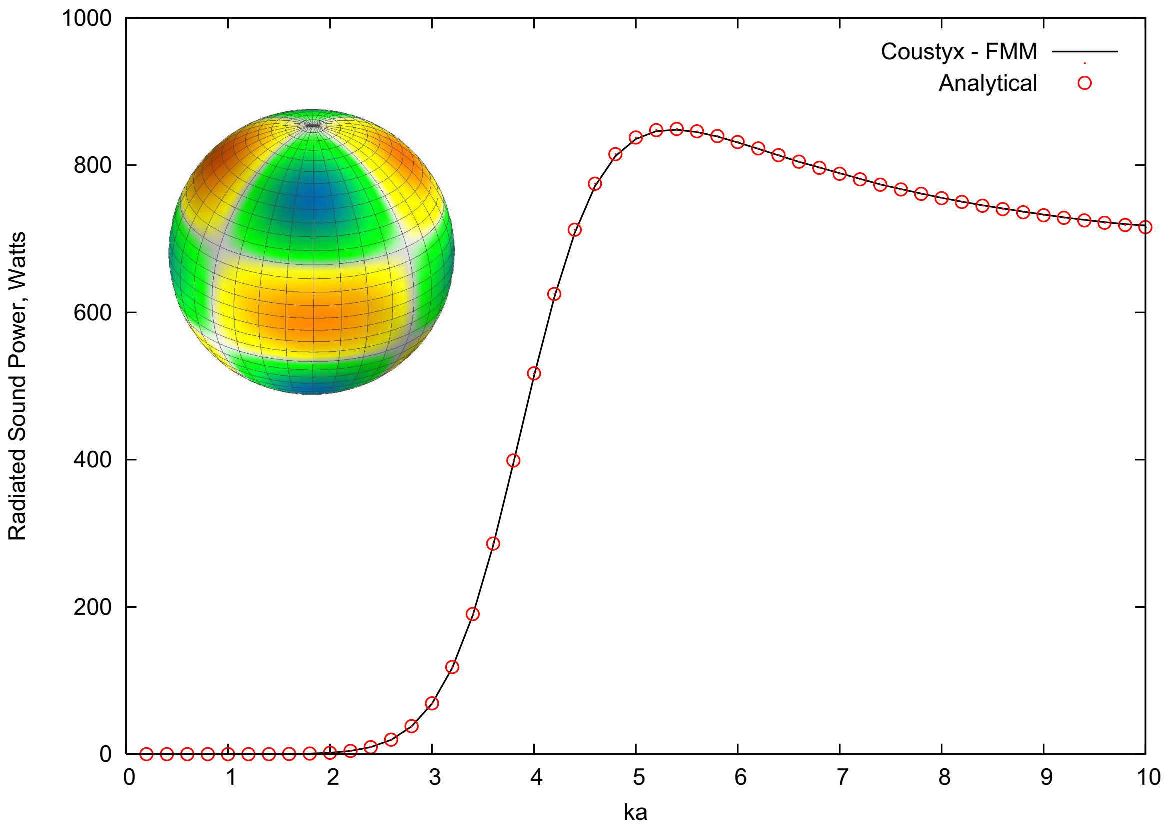

Figure 3: Radiated sound power comparisons for a sphere with spherical harmonic excitation.