Linear Duct with a Movable Wall - Exact Solution

In this example problem, the sound field inside a linear duct is studied. The wall on the right hand side is movable and is supported by a spring and damper. In addition, the right wall is acoustically rigid. Vibration excitation is provided by shaking the left wall. Here, we describe the steps in preparing the model, and compare the solution from Coustyx with the exact analytical solution. We came across this example in a paper by Suzuki [1] and created a Coustyx model to try it out. The derivation of the analytical solution based on structure mobility, although straightforward, is new.

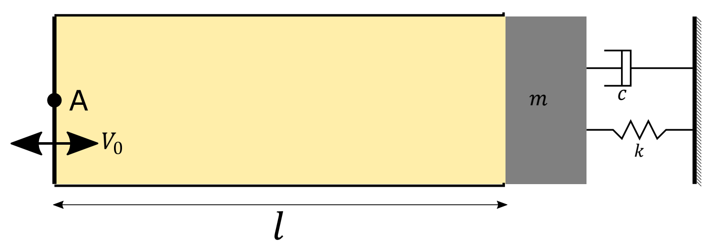

Figure 1: Linear duct with a movable wall.

1 Model Preparation

We have a duct of length $l = 0.34\,\text{m}$ and cross section $S = 0.08\,\text{m} \times 0.08\,\text{m}$. The BEM model is created by importing the surface mesh of the duct, and specifying the fluid domain to be bounded and on the interior of the mesh. The side of the surface mesh on the which the fluid domain of interest lies, is obtained by first inspecting the orientation of the surface BEM mesh. If the element normals are pointing to the outside, we specify that the fluid domain is on the *negative* side of the mesh, as we are interested in the interior problem.

The structure model corresponding the right plate (`right_plate`) and the rigid body mode at 400.10 Hz are imported from a Nastran `op2` file. Further, the coupling type of the structure is set as *Coupled* as we have to consider the two way fluid-structure coupling at $z = l$.

The next task is to apply the correct boundary conditions to various faces of the BEM mesh. On the left face, we specify a *uniform normal velocity boundary* of $v_n = -0.001\,\text{m/s}$. This corresponds to $v_z = 1\,\text{mm/s}$, as the domain normal is oriented in the negative z direction. Zero normal velocity (rigid boundary condition) is specified on the side walls. On the right face at $z = l$, a structure velocity boundary condition named `z1str` is specified. This boundary condition couples the acoustic domain with the structure on this boundary surface.

2 Analytical Solution

2.1 Driving point mobility of the SDOF System

We assume time harmonic variation of $e^{-i\omega t}$ where ω is the excitation frequency. Let $f(t) = Fe^{-i\omega t}$ be the applied force and $x(t) = Xe^{-i\omega t}$ be the deflection of the mass m.

$$m\ddot{x} + c\dot{x} + k = f \tag{1}$$

$$(-m\omega^2 - ic\omega + k)X = F \tag{2}$$

$$\frac{X}{F} = \frac{1}{(-m\omega^2 - ic\omega + k)} \tag{3}$$

The driving point mobility $Y = V_m/F = -i\omega X/F$ is given as

$$\begin{aligned} Y = \frac{V_m}{F} &= \frac{-i\omega}{(-m\omega^2 - ic\omega + k)} \\ Y(\omega) &= \frac{-ir}{(m\omega_n)((1 - r^2) - i(2\zeta r))} \end{aligned} \tag{4}$$

where $r = \omega/\omega_n$ is the ratio of the excitation frequency ω to the natural frequency ω_n of the SDOF system. In the SDOF system used in this example, mass $m = 0.01$ Kg, stiffness $k = 6.32 \times 10^4$N/m and damping $c = 0.503$N s/m. This corresponds to an in vacuo natural frequency of 400 Hz and damping ratio of $\zeta = 0.01$.

2.2 Acoustic field inside the duct

We can assume the sound field inside the duct to be of the form (In this derivation we use the coordinate x to be the position coordinate in the duct. In the Coustyx model the duct is along the z direction.)

$$p(x) = Ae^{ikx} + Be^{-ikx} \tag{5}$$

The expression for velocity in the x-direction is

$$v_x(x) = \frac{A}{Z_0}e^{ikx} - \frac{B}{Z_0}e^{-ikx} \tag{6}$$

The coefficients A and B are obtained so that the prescribed velocity boundary condition at $x = 0$ and boundary condition at $x = l$ are satisfied exactly.

The velocity boundary condition at $x = 0$ is $v_x(0) = V_0$. This yields

$$\begin{aligned} V_0 &= \frac{A}{Z_0} - \frac{B}{Z_0} \\ V_0 Z_0 &= A - B \end{aligned} \tag{7}$$

Consider the boundary condition at $x = l$. At this boundary, the acoustic particle velocity $v_x(l)$ must be same as the structure velocity V_m. It can be written as

$$v_x(l) = V_m \tag{8}$$

However, the velocity of the mass V_m is obtained using the driving point mobility $Y(\omega)$ as

$$V_m = p(l)SY(\omega) \tag{9}$$

where S is the cross-sectional area over which the sound pressure loading $p(l)$ acts on the structure.

From Equation 8 and Equation 9, the boundary condition at $x = l$ can be expressed as

$$p(l)SY(\omega) - v_x(l) = 0 \tag{10}$$

The system of equations for the unknown coefficients A and B are obtained from Equation 7 and Equation 10 as

$$\begin{bmatrix} 1 & -1 \\ e^{ikl}(SYZ_0 - 1) & e^{-ikl}(SYZ_0 + 1) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} V_0 Z_0 \\ 0 \end{Bmatrix} \tag{11}$$

Solving for A and B yields

$$\left\{ \begin{array}{c} A \\ B \end{array} \right\} = \frac{V_0 Z_0}{\Delta} \left\{ \begin{array}{c} e^{-ikl}(SYZ_0 + 1) \\ -e^{ikl}(SYZ_0 - 1) \end{array} \right\} \tag{12}$$

The expressions for the pressure field inside the duct is given as

$$p(x) = \frac{V_0 Z_0}{\Delta} \left\{ e^{-ik(l-x)}(SYZ_0 + 1) - e^{ik(l-x)}(SYZ_0 - 1) \right\}$$

$$p(x) = V_0 Z_0 \left\{ \frac{e^{-ik(l-x)}(SYZ_0 + 1) - e^{ik(l-x)}(SYZ_0 - 1)}{e^{-ikl}(SYZ_0 + 1) + e^{ikl}(SYZ_0 - 1)} \right\} \tag{13}$$

The expressions for the velocity field inside the duct is given as

$$v_x(x) = \frac{V_0}{\Delta} \left\{ e^{-ik(l-x)}(SYZ_0 + 1) + e^{ik(l-x)}(SYZ_0 - 1) \right\}$$

$$v_x(x) = V_0 \left\{ \frac{e^{-ik(l-x)}(SYZ_0 + 1) + e^{ik(l-x)}(SYZ_0 - 1)}{e^{-ikl}(SYZ_0 + 1) + e^{ikl}(SYZ_0 - 1)} \right\} \tag{14}$$

Substituting $x = 0$ in Equation 14 yields $v_x(0) = V_0$, as should be expected.

2.3 Comparison with exact Solution

We ran Coustyx on this model from 20 Hz to 1000 Hz in 20 Hz increments. The sound pressure at the vibrating left wall ($z = 0$) from Coustyx is compared with the exact solution (Equation 13) in Figure 2. It is evident that there is an excellent agreement between the two. The maximum relative error was 6.22% at 240 Hz when the exact pressure at $z = 0$ is minimum. Notice that the structural acoustic coupling decreased the structure resonant frequency from 400 Hz (in vacuo mode)to 380 Hz, while the first in vacuo acoustic mode increased from 500 Hz to 520 Hz.

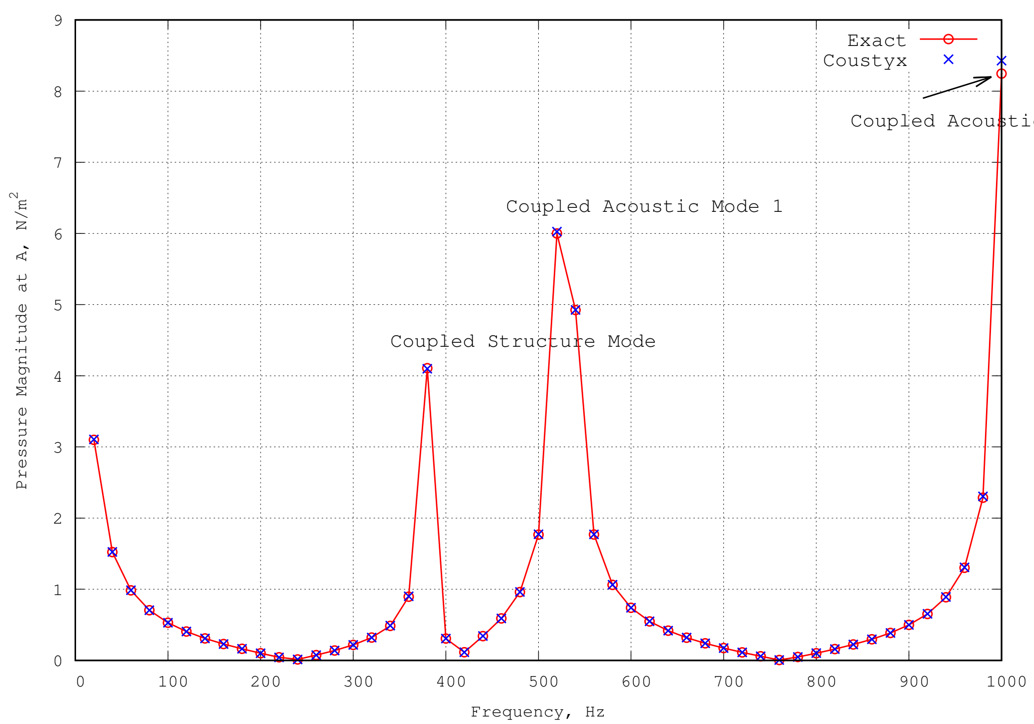

Figure 2: Frequency response of the sound pressure at Point A, comparison with exact solution.

References

[1] S. Suzuki, S. Maruyama, and H. Ido. Boundary element analysis of cavity noise problems with complicated boundary conditions. *Journal of Sound and Vibration*, 130(1):79–91, 1989.