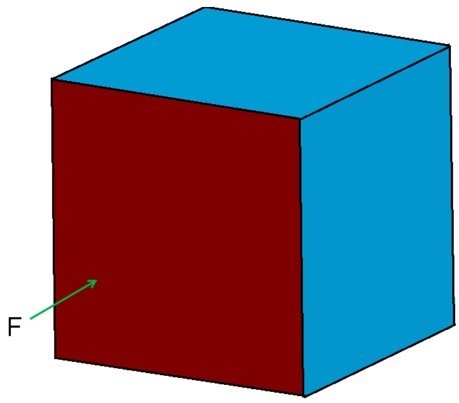
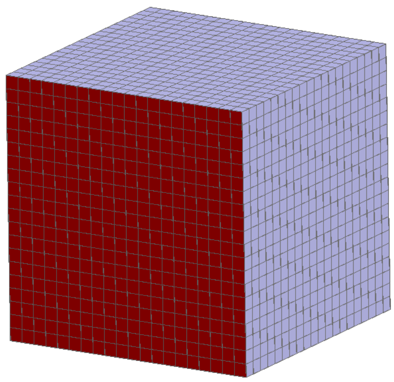
Rectangular panel backed by a closed cavity

1 Introduction

This example demonstrates the structural-acoustic coupling between an elastic rectangular panel backed by a closed rectangular acoustic cavity. Structural-acoustic coupling is typically modeled when the response of a structure is influenced by the acoustic pressure on its surface and vice versa. Typically thin shell structures are readily excited by acoustic pressure when the ambient density of the medium is comparable to the structural density. Modeling structural-acoustic coupling for such cases is important as the coupled system behaves entirely different from the uncoupled structure or uncoupled acoustic systems.
An analytical solution based on the series expansion derived by Pretlove [1] is used to check the accuracy of Coustyx results.



(a) Problem Schematic



(b) Boundary Element Model

Figure 1: Rectangular panel backed by a closed cavity.

2 Problem Statement

A schematic of the rectangular panel backed by the cavity is shown in Figure 1. The cavity is backed by the flexible panel on one side and by rigid walls on all other sides. The panel is simply supported on its edges. The panel has dimensions 1 m x 1 m and thickness $t = 0.01\,\mathrm{m}$. The cavity has dimensions 1 m x 1 m x 1 m. The panel is assumed to be made of steel with young's modulus $E = 210\,\mathrm{GPa}$, density $\rho_s = 7900\,\mathrm{kg/m^3}$ and poisson's ratio $\nu = 0.3$. The fluid medium inside the cavity is water with a sound speed of $c = 1481\,\mathrm{m/s}$ and an ambient density of $\rho_w = 1000\,\mathrm{Kg/m^3}$. A point force of $f = 1N$ is applied at location (0.2,0.3) on the panel to excite the system. We are interested in computing the response at the same location with the two-way structural-acoustic interaction taken into account.

3 Analytical Solution

A series solution developed by Pretlove [1] is used for analytical solution. The solution is rederived here for completeness. Note that the equations in the paper are modified to suit our time-dependence convention $e^{-j\omega t}$.
Let us consider a cavity of dimensions a x b x c. The panel on the side of the cavity at $z = c$ is assumed to be flexible and the rest of the side walls are assumed to be rigid.

The Helmholtz wave equation for the acoustic cavity is given by

$$\left[\nabla^2 + k^2\right] p(x,y,z) = 0 \tag{1}$$

where $p(x,y,z)$ is the acoustic pressure field, and $k = \frac{\omega}{c}$ is the wave number.
Employing separation of variables we look for solutions of the form,

$$p(x,y,z) = X(x)Y(y)Z(z) \tag{2}$$

The boundary conditions the wave equation needs to satisfy are,

$$\text{at x} = 0, \text{a} : \ \frac{\partial p(x,y,z)}{\partial x} = \frac{dX(x)}{dx} = 0 \tag{3}$$

$$\text{at y} = 0, \text{b} : \ \frac{\partial p(x,y,z)}{\partial y} = \frac{dY(y)}{dy} = 0 \tag{4}$$

$$\text{at z} = 0 : \ \frac{\partial p(x,y,z)}{\partial z} = \frac{dZ(z)}{dz} = 0 \tag{5}$$

$$\text{at z} = \text{c} : \ \frac{\partial p(x,y,z)}{\partial z} = X(x)Y(y)\frac{dZ(z)}{dz} = j\rho_w \omega \dot{w}_p \tag{6}$$

where $\dot{w}_p = -j\omega w_p$ is the velocity and w_p is the transverse vibration of the flexible panel, ρ_w is ambient density of fluid, ω is the frequency of analysis, and $j = \sqrt{-1}$
From the rigid boundary conditions at $x = 0, a$, and $y = 0, b$, the general solution takes the form,

$$p(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right) Z(z) \tag{7}$$

Substituting Equation 7 in Equation 1 we obtain

$$\frac{d^2 Z_{nm}(z)}{dz^2} - \mu_{nm}^2 Z_{nm} = 0 \tag{8}$$

where,

$$\mu_{nm}^2 = \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right] - \left(\frac{\omega}{c}\right)^2$$

The general solution for Equation 8 is of the form

$$Z_{mn}(z) = A\cosh\left(\mu_{nm} z\right) + B\sinh\left(\mu_{nm} z\right)$$

Applying the boundary conditions at $z = 0, c$ from Equation 5 and Equation 6 we obtain

$$B = 0$$

and,

$$j\rho_w \omega \dot{w}_p = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\mu_{nm} A_{nm}\sinh\left(\mu_{nm} c\right) \tag{9}$$

Now let us look at the general expression for plate deflection. Since the flexible panel at $z = c$ is simply supported, the vibration of the panel is given in terms of plate modes as follows:

$$w_p = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} W_{rs}\sin\left(\frac{r\pi x}{a}\right)\sin\left(\frac{s\pi y}{b}\right) \tag{10}$$

where W_{rs} is the modal contribution of the mode at (r, s) to the total response.
The double sines in the Equation 10 could be expanded as a series sum of double cosines using the following expressions.

$$\sin\left(\frac{r\pi x}{a}\right)\sin\left(\frac{s\pi y}{b}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \alpha_{nm}^{rs}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$$

where the coefficient α_{nm}^{rs} is

$$\alpha_{00}^{rs} = \frac{4}{\pi^2 rs}$$

$$\alpha_{n0}^{rs} = -\frac{8r}{\pi^2 s(n^2 - r^2)}$$

$$\alpha_{n0}^{rs} = -\frac{8s}{\pi^2 r(m^2 - s^2)}$$

$$\alpha_{nm}^{rs} = \frac{16rs}{\pi^2(n^2 - r^2)(m^2 - s^2)}; \text{n, m} \neq 0$$

For a plate mode at (r, s), Equation 9 reduces to

$$\rho_w \omega^2 W_{rs} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{nm}^{rs} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \mu_{nm} A_{nm} \sinh\left(\mu_{nm} c\right)$$

Therefore,

$$A_{nm} = \frac{\rho_w \omega^2 W_{rs} \alpha_{nm}^{rs}}{\mu_{nm} \sinh\left(\mu_{nm} c\right)}$$

Hence the acoustic pressure inside the cavity due to the (r, s) plate mode is

$$p(x, y, z) = \rho_w \omega^2 W_{rs} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha_{nm}^{rs}}{\mu_{nm}} \frac{\cosh\left(\mu_{nm} z\right)}{\sinh\left(\mu_{nm} c\right)} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \tag{11}$$

Let us now examine the equation of motion for the forced vibration of the panel. It is given by

$$\rho_s t \ddot{w}_p + D\left[\frac{\partial^4 w_p}{\partial x^4} + 2\frac{\partial^4 w_p}{\partial x^2 \partial y^2} + \frac{\partial^4 w_p}{\partial y^4}\right] = F^{ext}(x, y) + F(x, y) \tag{12}$$

where $D = \frac{Et^3}{12(1-\nu^2)}$ is the bending stiffness of the plate, $F^{ext}(x, y)$ is the sum of all external applied forces, and $F(x, y)$ is the acoustic pressure loading. Here the plate is applied with a point force of amplitude F_0 at (x_0, y_0). Therefore, $F^{ext}(x, y) = F_0 \delta(x - x_0, y - y_0)$, where δ is the Dirac delta function.
Using Equation 10 and the orthogonality of plate modes, the equation of motion for a $(r, s)^{th}$ plate mode reduces to

$$-\omega^2 \rho_s t \frac{ab}{4} W_{rs} + D\frac{ab}{4}\left[\left(\frac{r\pi}{a}\right)^2 + \left(\frac{s\pi}{b}\right)^2\right]^2 W_{rs} = F_{rs}^{ext} + F_{rs} \tag{13}$$

where the external modal force (F_{rs}^{ext}) and acoustic modal force (F_{rs}) are

$$F_{rs}^{ext} = F_0 \sin\left(\frac{r\pi x_0}{a}\right) \sin\left(\frac{s\pi y_0}{b}\right)$$

$$F_{rs} = \int_{x=0}^{a} \int_{y=0}^{b} p_{back} \sin\left(\frac{r\pi x}{a}\right) \sin\left(\frac{s\pi y}{b}\right) dxdy$$

and p_{back} is the acoustic back pressure on the plate.
We recognize that the acoustic loading at (r, s) plate mode includes contributions from the back pressures $p_{back}^{r's'}$ at other modes (r', s') as well. Using Equation 11 for estimating the back pressure $p_{back}^{r's'}$ and applying the orthogonality conditions, the acoustic loading at (r, s) mode due to the back pressure at (r', s') mode is given by

$$F_{rs}^{r's'} = \rho_w \omega^2 W_{rs} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha_{nm}^{rs} \alpha_{nm}^{r's'}}{\mu_{nm}} \coth\left(\mu_{nm} c\right) \int_{x=0}^{a} \cos^2\left(\frac{n\pi x}{a}\right) dx \int_{y=0}^{b} \cos^2\left(\frac{n\pi y}{b}\right) dy \tag{14}$$

Finally, the equation of motion at (r, s) mode can be rewritten as

$$-\omega^2 M_{rs} W_{rs} + K_{rs}^{plate} W_{rs} + \sum_{r's'} K_{rs}^{r's'} W_{r's'} = F_0 \sin\left(\frac{r\pi x_0}{a}\right) \sin\left(\frac{s\pi y_0}{b}\right) \tag{15}$$

and,

$$M_{rs} = \rho_s t \frac{ab}{4}$$

$$K_{rs}^{plate} = D\frac{ab}{4}\left[\left(\frac{r\pi}{a}\right)^2 + \left(\frac{s\pi}{b}\right)^2\right]^2$$

$$K_{rs}^{r's'} = -\rho_w \omega^2 W_{rs} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha_{nm}^{rs}\alpha_{nm}^{r's'}}{\mu_{nm}} \coth\left(\mu_{nm} c\right) \int_{x=0}^{a} \cos^2\left(\frac{n\pi x}{a}\right) dx \int_{y=0}^{b} \cos^2\left(\frac{n\pi y}{b}\right) dy$$

where M_{rs}, K_{rs}^{plate} are the modal mass and modal stiffness matrices of the plate; $K_{rs}^{r's'}$ is the cross stiffness terms due to the structural-acoustic coupling.
To obtain a solution to the above equation (Equation 15), we limit the number of modes in the infinite series and form system matrix that is inverted to solve for the unknown modal coefficients W_{rs}. Once we obtain W_{rs}, we can then easily compute the panel vibration or acoustic pressure inside the cavity using Equation 10 and Equation 11 respectively.

4 Coustyx Solution

The structural-acoustic coupling in Coustyx is modeled by combining the finite element method for structural analysis and the boundary element method for acoustic analysis. Instead of using the finite element mass and stiffness matrices, Coustyx formulations use the normalized invacuo vibration modes (ortho-normalized with respect to the finite element mass matrix) and eigen frequencies for modeling the structure. The invacuo modes and natural frequencies are computed from the modal analysis performed using any external FE packages.
The steps followed to model the two-way structural-acoustic interaction are:

- **Perform finite element modal analysis**. Finite element modal analysis is performed to compute modes and eigen frequencies of the structure in vacuum (in the absence of any fluid medium). These invacuo modes are ortho-normalized with respect to the finite element mass matrix before importing into Coustyx.
- **Load modes and perform coupled analysis in Coustyx**. The invacuo modes computed in the previous step are loaded into Coustyx. The coupled analysis in Coustyx is performed by applying structure velocity boundary condition on the boundary element mesh with the structure tagged as coupled.

4.1 Finite element analysis

We performed modal analysis on the simply supported plate using external finite element software to estimate invacuo modes and natural frequencies. The finite element model of the plate has 20 x 20 quadratic shell elements (Figure 2). Table 4.1 compares the first few natural frequencies computed from closed form analytical solution and finite element analysis. Figure 3 shows the first six free vibration mode shapes for the plate.

4.2 Coupled Model in Coustyx

To setup Coustyx model, we imported the plate structure mesh and the invacuo modes computed from the finite element modal analysis performed earlier. The plate structure is tagged as Coupled using Coupling type options. The external excitation is applied as a nodal force of value 1 at the node located at (0.2,0.3). We added small modal damping for all the modes.

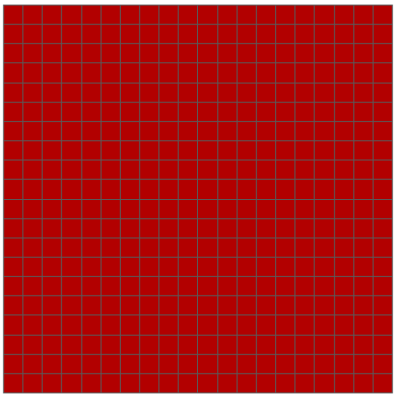
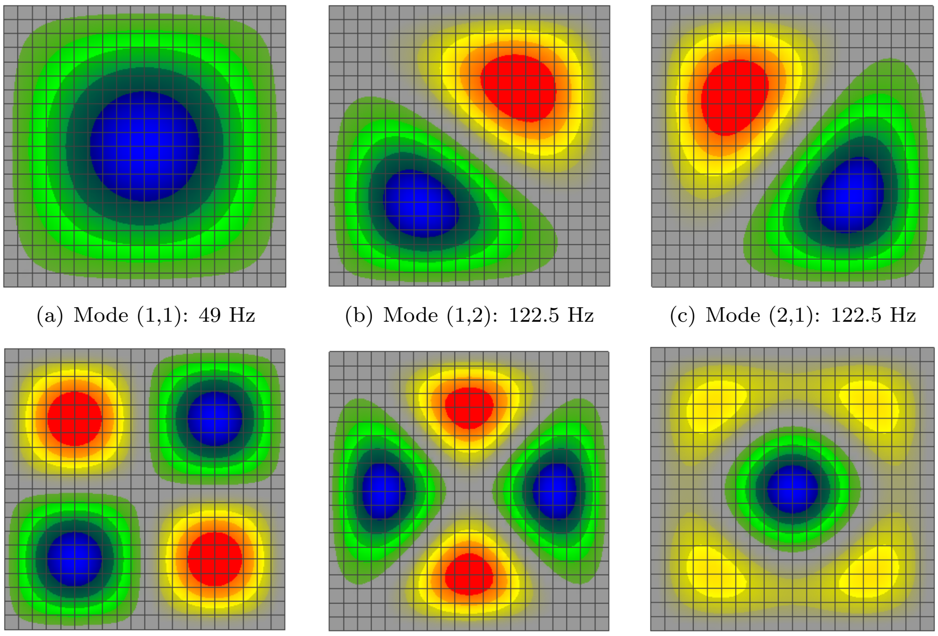


Figure 2: Panel finite element mesh.



(a) Mode (1,1): 49 Hz

(b) Mode (1,2): 122.5 Hz

(c) Mode (2,1): 122.5 Hz

(d) Mode (2,2): 196 Hz

(e) Mode (1,3): 245 Hz

(f) Mode (3,1): 245 Hz

Figure 3: Invacuo mode shapes for the first six free vibration modes of a simply supported plate.

Table 1: Invacuo natural frequencies for a simply supported plate

Mode (r,s)	Analytical (Hz)	FEA (Hz)
(1,1)	49.02	49.00
(1,2)	122.54	122.43
(2,1)	122.54	122.43
(2,2)	196.06	195.78
(1,3)	245.08	244.65
(3,1)	245.08	244.65
(2,3)	318.60	317.87
(3,2)	318.60	317.87
(1,4)	416.63	415.47
(4,1)	416.63	415.47
(3,3)	441.14	439.74
(2,4)	490.16	488.50
(4,2)	490.16	488.50
(3,4)	612.69	610.06
(4,3)	612.69	610.06

The boundary element mesh of the cavity used for Coustyx model is shown in Figure 1. The side of the cavity at $z = 0$ is applied with the plate structure velocity boundary condition. All other sides are applied with rigid boundary conditions, $v_n = 0$.
Analysis is carried out by running the Analysis Sequences defined in the Coustyx model named `Run Validation - FMM` . An Analysis Sequence stores all the user inputs specified for an analysis, such as boundary integral formulation type, frequency range and spacing, solution method, along with various requested outputs.

5 Results and Discussion

Figure 4 shows the frequency response (displacement amplitude in z direction) at the location (0.2,0.3) from both Coustyx and analytical solutions. The comparisons show that Coustyx predicts the frequency response of the coupled system very accurately. We note that the resonant peaks for the coupled system occur at frequencies entirely different from the plate invacuo natural frequencies or acoustic room modes.
The vibration modes of the panel near each of the resonant peaks is shown in Figure 4. For invacuo plate modes with non-zero average flux at the plate interface the acoustic cavity have the effect of added stiffness to the system. This makes these modes appear at much higher frequencies in the coupled system. For example, the invacuo plate mode(1,1) at 49 Hz does not appear in the frequency range of 10-250 Hz at all. Whereas for the invacuo plate modes with zero average flux at the plate interface the acoustic cavity have the effect of added mass to the system. For example, the invacuo plate mode(2,1) at 122.5 Hz appears at lower frequency 61 Hz due to this effect. Other resonant peaks at 112, 134, 150 and 201 Hz have mode shapes that correspond to invacuo plate modes with zero average flux at the plate interface.

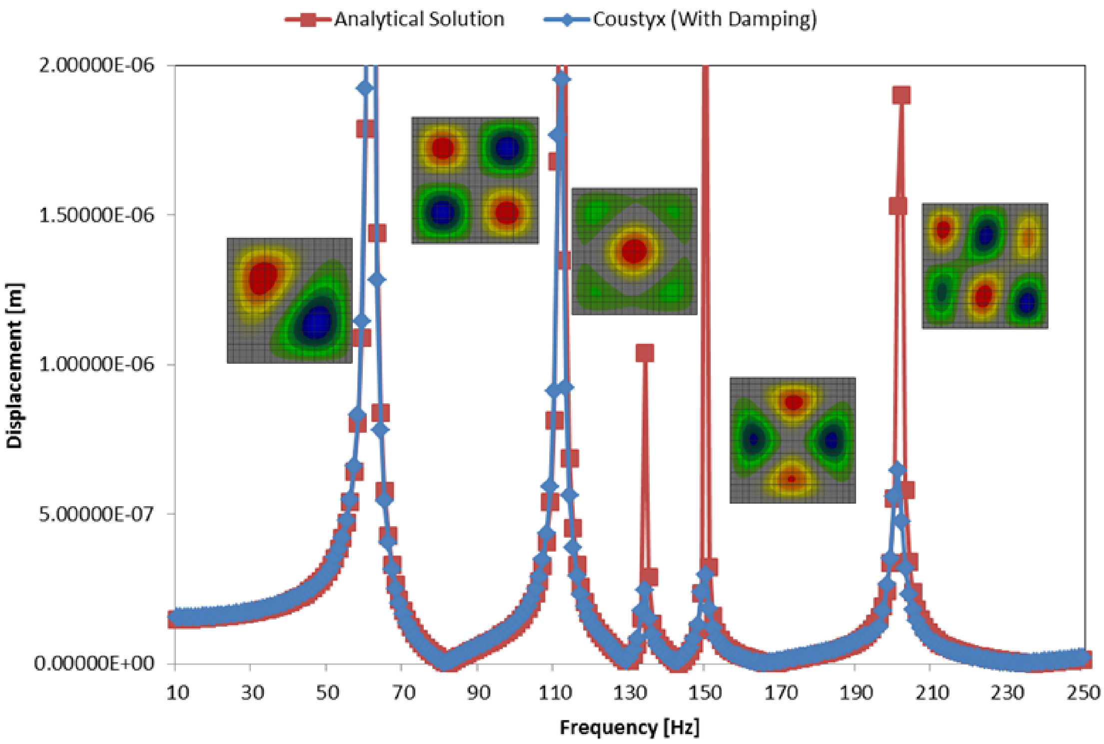


Figure 4: Frequency response of the panel coupled to the acoustic cavity.

References

[1] A. J Pretlove. Free vibrations of a rectangular panel backed by a closed rectangular cavity. *Journal of Sound and Vibration*, 2:197–209, 1965.