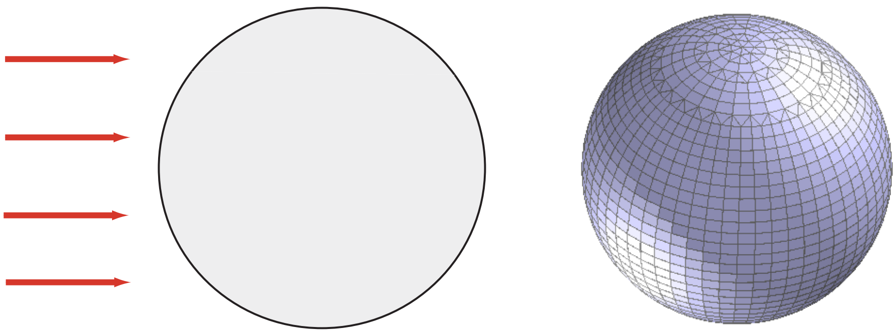
Plane Wave Scattering by an Elastic Spherical Shell

1 Introduction

In this example we will investigate the sound scattered by an elastic spherical shell impinged by an acoustic plane wave. We consider the two-way coupling between the acoustic pressure and the structure response to account for (a) the response of the shell to the surface pressure, and (b) the sound radiated by the vibrating shell. Modeling two-way coupling is important for many systems, especially for systems with thin elastic shell structures submerged in heavy fluids like water. An analytical solution presented in Junger and Feit [1] is used to validate the accuracy of Coustyx results.



(a) Problem Schematic (b) Boundary Element Model

Figure 1: Acoustic problem description.

2 Problem Statement

A schematic of the problem is shown in Figure 1. An elastic spherical shell with middle surface radius $a = 1\,\mathrm{m}$, thickness $h = 0.01\,\mathrm{m}$, and centered at the origin (0,0,0) is considered. A plane wave of amplitude $p_0 = 1$ propagating in the +z direction is scattered by the sphere. The fluid medium surrounding the sphere is assumed to be water with a sound speed of $c = 1481\,\mathrm{m/s}$ and an ambient density of $\rho = 1000\,\mathrm{Kg/m^3}$. The spherical shell is assumed to be made of steel with young's modulus $E = 210\,\mathrm{GPa}$, density $\rho_s = 7800\,\mathrm{kg/m^3}$ and poisson's ratio $\nu = 0.3$. We are interested in computing the sound scattered by the elastic sphere.

3 Analytical Solution

Analytical solution presented in Junger and Feit [1] is described here briefly for completeness. For greater details please refer to their work. Note: we assume $e^{-i\omega t}$ convention through out this article.

An incident plane wave of amplitude p_0 propagating in +z direction is given by

$$p_{inc} = p_0 e^{ikz} \tag{1}$$

The plane wave could be represented as a series sum of spherical harmonics as follows,

$$p_{inc} = p_0 \sum_{l=0}^{N} (2l+1) i^l P_l(\cos\theta) j_l(kr) \tag{2}$$

where j_l is the spherical Bessel function, P_l is the Legendre function of order l, and $i = \sqrt{-1}$.

Let p_{total} be the total sound pressure in the domain,

$$p_{total} = p_{inc} + p_{se} \tag{3}$$

where p_{inc} is the incident pressure and p_{se} is the pressure scattered by the elastic sphere.

Let us further consider that the scattered pressure by the elastic sphere p_{se} is a sum of the scattered pressure by a rigid sphere $p_{s\infty}$, and the radiated pressure due to shell vibration p_r, i.e.,

$$p_{se} = p_{s\infty} + p_r \tag{4}$$

The boundary condition on the sphere is

$$\frac{\partial p_{total}}{\partial r} = i\rho\omega\dot{w} \tag{5}$$

where $\dot{w}$ is the radial velocity of the spherical shell, and ω is the frequency of excitation in rad/s.

By definition, on the surface of the sphere the "rigid-body" scattered pressure $p_{s\infty}$ and the incident pressure p_{inc} are related as follows,

$$\frac{\partial p_{inc}}{\partial r} + \frac{\partial p_{s\infty}}{\partial r} = 0 \tag{6}$$

From Equation 5 and Equation 6 we can deduce that the radiated pressure p_r is directly related to the vibration of the spherical shell as

$$\frac{\partial p_r}{\partial r} = i\rho\omega\dot{w} \tag{7}$$

The pressure scattered by a rigid sphere impinged by a plane wave is given by [2]

$$p_{s\infty} = -p_0 \sum_{l=0}^{N} (2l+1) i^l P_l(\cos\theta) \frac{j'_l(ka)}{h'_l(ka)} h_l(kr) \tag{8}$$

where h_l is the spherical Hankel function of the first kind of order l, and $' \equiv \frac{\partial}{\partial r}$.

Following the arguments in Junger and Feit [1], we consider only the sum of the incident and "rigid-body" scattered pressures ($p_{inc}+p_{s\infty}$) as the surface pressure excitation and treat the radiation loading as part of the coefficient (LHS) matrix through radiation impedance. Thus, the surface pressure excitation on the spherical shell is

$$p_{ext} = p_{inc} + p_{s\infty} = p_0 \sum_{l=0}^{N} \frac{(2l+1) i^{l+1}}{(ka)^2 h'_l(ka)} P_l(\cos\theta) = \sum_{l=0}^{N} p_l P_l(\cos\theta) \tag{9}$$

The forced velocity response of the spherical shell due to the excitation in Equation 9 is given by

$$\dot{w} = -\sum_{l=0}^{N} \frac{p_l P_l(\cos\theta)}{(Z_l + z_l)} \tag{10}$$

where Z_l is the *invacuo* modal impedance of the spherical shell, and z_l is the modal specific acoustic impedance.

The *invacuo* modal impedance of the spherical shell is given below. Note that these equations are only applicable for axisymmetric motions.

$$Z_l = -\frac{i\rho_s c_p}{\Omega} \cdot \frac{h}{a} \cdot \frac{\left[\Omega^2 - \left(\Omega_l^{(1)}\right)^2\right]\left[\Omega^2 - \left(\Omega_l^{(2)}\right)^2\right]}{[\Omega^2 - (1+\beta^2)(\nu + \lambda_l - 1)]} \tag{11}$$

where $\Omega_l^{(1)}$, $\Omega_l^{(2)}$ are the nondimensional natural frequencies of the spherical shell obtained by solving Equation 12, $\lambda_l = l(l+1)$, c_p is the velocity of compressional waves in the structure as given in Equation 13, $\Omega = \omega \frac{a}{c_p}$ is dimensionless driving frequency, and,

$$\beta^2 = \frac{h^2}{12a^2}$$

The nondimensional natural frequencies of the spherical shell (with axisymmetric motions assumption) are computed by solving the equation

$$\begin{aligned}\Omega^4 - \left[1 + 3\nu + \lambda_l - \beta^2(1 - \nu - \lambda_l^2 - \nu\lambda_l)\right]\Omega^2 + (\lambda_l - 2)(1 - \nu^2) \\ +\beta^2\left[\lambda_l^3 - 4\lambda_l^2 + \lambda_l(5 - \nu^2) - 2(1 - \nu^2)\right] = 0\end{aligned} \tag{12}$$

The compressional wave speed in the structure is given by

$$c_p = \sqrt{\frac{E}{(1 - \nu^2)\rho_s}} \tag{13}$$

The modal specific acoustic impedance is given by

$$z_l = i\rho c\frac{h_l(ka)}{h'_l(ka)} \tag{14}$$

The radiated pressure due to the spherical shell vibration in Equation 10 is given by

$$p_r = p_0 \sum_{l=0}^{N} \frac{(2l+1)i^l\rho c}{(Z_l + z_l)\left[kah'_l(ka)\right]^2} P_l(\cos\theta)h_l(kr) \tag{15}$$

The total scattered pressure by an elastic spherical shell is obtained from Equation 4, $p_{se} = p_{s\infty} + p_r$, where the "rigid-body" scattered pressure $p_{s\infty}$ is from Equation 8, and the radiated pressure is from Equation 15.

4 Coustyx Solution

The structural-acoustic coupling in Coustyx is modeled by combining the finite element method for structural analysis and the boundary element method for acoustic analysis. Instead of using the finite element mass and stiffness matrices, Coustyx formulations use the normalized invacuo vibration modes (ortho-normalized with respect to the finite element mass matrix) and eigen frequencies for modeling the structure. The invacuo modes and natural frequencies are computed from the modal analysis performed using any external FE packages.

The steps followed to model the two-way structural-acoustic interaction are:

- **Perform finite element modal analysis**. Finite element modal analysis is performed to compute modes and eigen frequencies of the structure in vacuum (in the absence of any fluid medium). These invacuo modes are ortho-normalized with respect to the finite element mass matrix before importing into Coustyx.
- **Load modes and perform coupled analysis in Coustyx**. The invacuo modes computed in the previous step are loaded into Coustyx. The coupled analysis in Coustyx is performed by applying structure velocity boundary condition on the boundary element mesh with the structure tagged as coupled.

4.1 Finite element analysis

We performed modal analysis on unconstrained sphere using an external finite element software to estimate invacuo modes and natural frequencies. The first few natural frequencies along with their degeneracy in paranthesis are 0 Hz (6), 606.85 Hz (3), 606.88 Hz (2), 718.84 Hz (3), 718.85 Hz (2), etc. The first six modes with zero frequencies belong to the rigid body motions of the sphere.

4.2 Coupled Model in Coustyx

To setup Coustyx model, we imported the sphere mesh and the invacuo modes computed from the finite element modal analysis performed earlier. The sphere structure is tagged as Coupled using Coupling type options. A small amount of modal damping is added to all the modes.

The boundary element mesh of the sphere used for Coustyx model is shown in Figure 1. A structure velocity boundary condition is applied to the entire sphere. A plane wave of amplitude $p_0 = 1$ is defined at the origin (0,0,0) in the +z direction.

Coustyx Direct BE method is used to solve the acoustic problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Field point solutions are then computed from the surface solutions. For this problem we assume the coupling between the structure and the exterior acoustic domain only.

5 Results and Discussion

Figure 2 shows the nondimensional scattered pressure at r=10 m and $\theta = 0$ from both Coustyx and analytical solutions. The comparisons show that Coustyx predicts the response of the coupled system very accurately. The resonance peaks at 280, 360, and 410 Hz correspond to the natural frequencies of the submerged elastic spherical shell (refer to Table 9.1 in [1]).

Figure 3 shows angular distribution of the scattered pressure field by the elastic sphere at r=10m at resonant and nonresonant frequencies of 280 and 500 Hz. The quantities are plotted versus the polar angle θ. $\theta = 180^o$ corresponds to the front end of the sphere with respect to the impinging plane wave. The comparisons between Coustyx and Analytical solution show very good agreement. Analytical solution for the sound scattered by a rigid sphere is also plotted for reference.

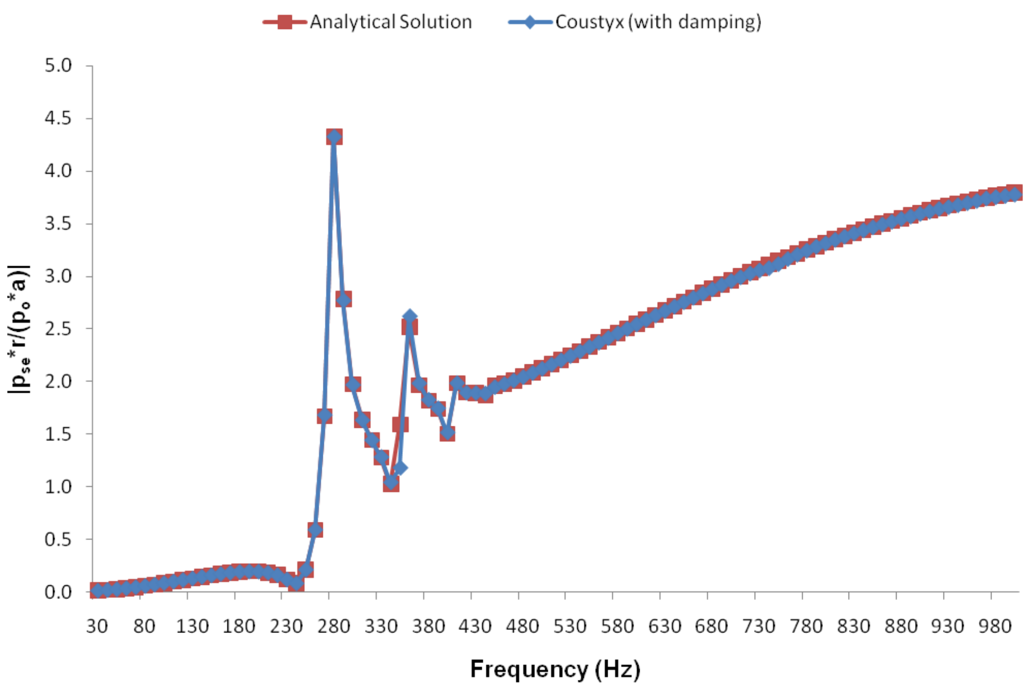
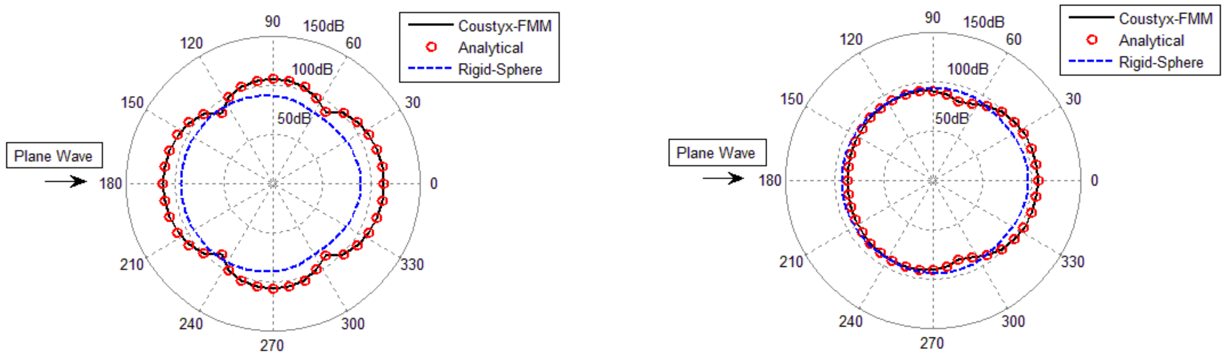


Figure 2: Back scattering by an elastic sphere.



(a) Frequency=280 Hz

(b) Frequency=500 Hz

Figure 3: Angular distribution of pressure field (in dB) scattered by an elastic sphere at different frequencies. Reference pressure for water $p_{ref} = 1\,\mu$Pa.

References

[1] Miguel C. Junger and David Feit. *Sound, Structures, and their Interaction*, chapter Elastic Scatterers and Wave Guides, Table 9.1, pages 228–234,282,352–358. MIT Press, Cambridge, Massachusetts, second edition, 1986.

[2] http://ansol.us/Products/Coustyx/Validation/. Plane wave scattering by a rigid sphere validation example.