Acoustic Field inside a Rigid Cylinder with a Point Source

1 Introduction

The main objectives of this Demo Model are to

- Demonstrate the ability of Coustyx to model a rigid cylinder with a point source using Coustyx MultiDomain model and solve for the acoustic field distribution inside the cylinder.
- Derive analytical solution using modal expansion method.
- Validate Coustyx software by comparing the results from Coustyx to the analytical solutions in the presence of acoustic sources.

2 Model description

We model a cylinder of radius $a = 1\,\mathrm{m}$ and length $L = 6\,\mathrm{m}$. The cylinder axis is parallel to the *z-axis* and the center of the bottom end of the cylinder coincides with the origin (as shown in Figure 1). The fluid medium inside the cylinder is air with mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$ and sound speed $c = 343 - i * 10\,\mathrm{m/s}$. A complex speed of sound introduces damping in the system. The imaginary part of the speed of sound should always be negative for a decaying sound wave. The wavenumber at a frequency ω is given as $k = omega/c$. A monopole source of unit volume velocity is introduced at (-0.2,-0.35,4) to simulate the point source in the cylinder. The cylinder is assumed to be rigid. The BE mesh of the cylinder is shown in Figure 1.

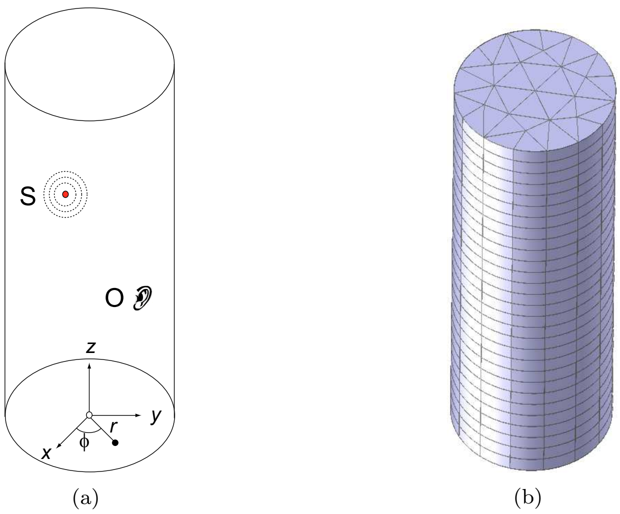

Figure 1: Rigid cylinder with a point source. (a) Acoustic problem, Note S–source, O–observation point; (b) Boundary element mesh.

3 Boundary conditions

In the Coustyx MultiDomain model, the rigid boundary condition is simulated by applying the boundary condition on the cylinder as "Uniform Normal Velocity" type with zero amplitude. That is, $v_n = 0$, where v_n is the particle normal velocity on the surface of the cylinder in the *Domain Normal* direction. Note that all boundary conditions in a MultiDomain model are defined with respect to the *Domain Normal*, which always points away from the domain of interest. For this example, the interior domain is the domain of interest; hence, domain normal is pointing away from the cylinder interior.

4 Analytical solution

We first compute modes for a rigid cylinder (of radius a and length L). These modes are then used in modal expansion to evaluate field point pressure at any point inside the cylinder.

4.1 Eigenvalue problem

Table 1: Eigenvalues of a rigid cylinder from the roots of $J'_m(k_{r,m}) = 0$, $k_{r,m}(n)$

$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
0.000000	1.841184	3.054237	4.201189	5.317553	6.415616
3.831706	5.331443	6.706133	8.015237	9.282396	10.519861
7.015587	8.536316	9.969468	11.345924	12.681908	13.987189
10.173468	11.706005	13.170371	14.585848	15.964107	17.312842
13.323692	14.863589	16.347522	17.788748	19.196029	20.575515
16.470630	18.015528	19.512913	20.972477	22.401032	23.803581
19.615859	21.164370	22.671582	24.144897	25.589760	27.010308
22.760084	24.311327	25.826037	27.310058	28.767836	30.202849
25.903672	27.457051	28.977673	30.470269	31.938539	33.385444
29.046829	30.601923	32.127327	33.626949	35.103917	36.560778

The eigen-function $\Psi(\mathbf{r}, n)$ satisfies the Helmholtz equation at any point $\mathbf{r}(r, \phi, z)$ inside the cylinder

$$\left[\nabla^2 + k_0^2(n)\right] \Psi(\mathbf{r}, n) = 0 \tag{1}$$

where $k_0^2(n)$ is the eigenvalue.
The eigen-function should also satisfy the rigid boundary conditions on the surface of the cylinder

$$\frac{\partial \Psi(\mathbf{r}, n)}{\partial \hat{\mathbf{n}}} = 0 \tag{2}$$

where $\hat{\mathbf{n}}$ is the surface normal.
In cylindrical coordinate system ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \tag{3}$$

To solve the eigenvalue problem, we assume that the eigen-function can be factored into a form $\Psi(\mathbf{r}, n) = R(r)e^{im\phi}Z(z)$. The Helmholtz equation is reduced to

$$\frac{1}{R(r)}\left[\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r}\right] - \frac{m^2}{r^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} + k_0^2 = 0 \tag{4}$$

Applying separation of variables, we obtain independent equations for the axial factor $Z(z)$ and the radial factor $R(r)$. The axial factor $Z(z)$ satisfies

$$\frac{\partial^2 Z(z)}{\partial z^2} + k_z^2 Z(z) = 0 \tag{5}$$

and also the rigid boundary conditions on both ends of the cylinder,

$$\frac{\partial Z}{\partial z} = 0, \; z = 0, \text{ and } L \tag{6}$$

Therefore, the axial factor $Z(z)$ is the solution to above equations, which is $Z(z) = cos(k_z z)$, and the eigen values $k_z = n_z\pi/L$ for $n_z = 0, 1, 2....$

The radial factor $R(r)$ satisfies

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \left(k_r^2 - \frac{m^2}{r^2}\right) R(r) = 0 \tag{7}$$

where $k_r^2 = k_0^2 - k_z^2$, and the rigid boundary conditions are

$$\frac{\partial R}{\partial r} = 0,\ r = a \tag{8}$$

The solution to the above equation is the cylindrical Bessel function $J_m(k_r r)$, where m is the order and k_r the eigenvalue.
The eigenvalues $(k_{r,m}(n))$ are the roots of the derivative of the cylindrical Bessel function at $r = a$, that is,

$$J_m'(k_{r,m}(n)a) = 0 \tag{9}$$

The above equation is solved numerically using Maple to find the eigenvalues. Table 4.1 shows the first ten eigenvalues for $m = 0, ..., 5$.
Therefore, the eigen-function for a rigid cylinder is given by

$$\Psi_{n_z,m}(n) = J_m(k_{r,m}(n)r)e^{im\phi}\cos(k_z z) \tag{10}$$

and the eigenvalues are $k_z = n_z\pi/2$ for $n_z = 0, 1, 2...$, and $k_{r,m}(n)$ given in Table 4.1.

4.2 Modal expansion

The acoustic field pressure p at any point $\mathbf{r}(r, \phi, z)$ (inside the cylinder) due to the presence of a point source at $\mathbf{r}_s(r_s, \phi_s, z_s)$ satisfies the Helmholtz equation

$$\nabla^2 p + k^2 p = \varpi\delta(\mathbf{r} - \mathbf{r}_s) \tag{11}$$

where the source strength $\varpi = ik\rho_o c\beta_o$, β_o is the volume velocity; $\delta(\mathbf{r} - \mathbf{r}_s)$ is the Dirac delta function.
The modal eigen-functions derived above form a complete set. Hence, the acoustic solution p inside the cylinder can be approximated as a linear combination of these eigen-functions.

$$\begin{aligned} p &= \sum_{n_z}\sum_m\sum_n C_{n_z,m,n}\Psi_{n_z,m}(\mathbf{r}, n) \\ &= \sum_{n_z}\sum_m\sum_n J_m(k_{r,m}(n)r)\cos(k_z z)\left[A_{n_z,m,n}\cos m\phi + B_{n_z,m,n}\sin m\phi\right] \end{aligned} \tag{12}$$

where $A_{n_z,m,n}$, $B_{n_z,m,n}$, and $C_{n_z,m,n}$ are mode participation coefficients.
We compute the mode participation coefficients by substituting Equation 12 into Equation 11, that is,

$$\sum_{n_z}\sum_m\sum_n \left[k^2 - k_o^2\right] J_m(k_{r,m}(n)r)\cos(k_z z)\left[A_{n_z,m,n}\cos m\phi + B_{n_z,m,n}\sin m\phi\right] = \varpi\delta(\mathbf{r} - \mathbf{r}_s) \tag{13}$$

We use the following properties to compute $A_{n_z,m,n}$ and $B_{n_z,m,n}$. Normalization integral for cylindrical Bessel functions (refer [1]),

$$\begin{aligned} \int_0^1 J_m^2(\alpha_m r)rdr &= \tfrac{1}{2\alpha_m^2}\left[\alpha_m^2 - m^2\right] J_m^2(\alpha_m),\ \text{for}\ \alpha_m \neq 0 \\ &= \tfrac{1}{2},\ \text{for}\ \alpha_m = 0 \end{aligned}$$

or,

$$\begin{aligned} N_l = \int_0^a J_m^2(k_{r,m}(n)r)rdr &= \tfrac{a^2}{2(k_{r,m}a)^2}\left[(k_{r,m}a)^2 - m^2\right] J_m^2(k_{r,m}a),\ \text{for}\ k_{r,m}a \neq 0 \\ &= \tfrac{1}{2},\ \text{for}\ k_{r,m}a = 0 \end{aligned}$$

and also,

$$\int_0^L \cos^2\frac{n_z\pi}{L}zdz = \begin{cases} L, & n_z = 0 \\ L/2, & n_z \neq 0 \end{cases}$$

$$\int_0^{2\pi} \cos^2 m\phi d\phi = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \neq 0 \end{cases}$$

$$\int_0^{2\pi} \sin^2 m\phi d\phi = \begin{cases} 0, & m = 0 \\ \pi, & m \neq 0 \end{cases}$$

To compute $A_{n_z,m,n}$, multiply both sides of Equation 13 with $J_m(k_{r,m}(n)r)\cos(k_z z)\cos(m\phi)$ and integrate over the entire volume. Using the properties of various functions described above, along with the properties of Dirac delta function, the coefficient $A_{n_z,m,n}$ is derived.

$$A_{n_z,m,n} = \varpi \frac{J_m(k_{r,m}(n)r_s)\cos(k_z z_s)\cos m\phi_s}{[k^2 - k_o^2]\, N_l 2\pi\varepsilon L\eta} \tag{14}$$

where $\varepsilon = 1$ for $m = 0$, and $\varepsilon = 1/2$ for $m \neq 0$; $\eta = 1$ for $n_z = 0$ and $\eta = 1/2$ for $n_z \neq 0$.
Similarly, $B_{n_z,m,n}$ is computed by multiplying both sides of Equation 13 with $J_m(k_{r,m}(n)r)\cos(k_z z)\sin(m\phi)$. Therefore,

$$B_{n_z,m,n} = \varpi\zeta \frac{J_m(k_{r,m}(n)r_s)\cos(k_z z_s)\sin m\phi_s}{[k^2 - k_o^2]\, N_l 2\pi L\eta} \tag{15}$$

where $\zeta = 0$ for $m = 0$, and $\zeta = 2$ for $m \neq 0$.
Thus the modal expansion for the pressure at an observation point $\mathbf{r}(r, \phi, z)$ inside the rigid cylinder in the presence of a point source at $\mathbf{r_s}(r_s, \phi_s, z_s)$ is,

$$p(r,\phi,z) = \sum_{n_z}\sum_{m}\sum_{n} \varpi \frac{J_m(k_{r,m}(n)r_s)\cos\left(\frac{n_z\pi}{L}z_s\right)}{[k^2 - k_o^2]\, N_l 2\pi\varepsilon L\eta} J_m(k_{r,m}(n)r)\cos\left(\frac{n_z\pi}{L}z\right)\cos\left(m(\phi - \phi_s)\right) \tag{16}$$

where $n_z = 0, 1, 2, ...$, $m = 0, 1, 2, ...$ and $n = 1, 2, ...$
If the position is in Cartesian coordinates, it can be transformed into cylindrical coordinates using the transformations given below:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \phi &= \arctan(y/x), 0 < \phi \leq 2\pi \\ z &= z \end{aligned} \tag{17}$$

5 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In the demo model, the analysis is performed at a frequency $f = 200Hz$ using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".
Coustyx MultiDomain model uses Direct BE method to solve the acoustic problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Field point solutions are then computed from the surface solutions.
Figure 2 shows comparisons of field point pressures computed from both Coustyx and analytical methods. The specified field points are located at (r, ϕ, z), where $r = 0.8485\,\text{m}, \phi = \pi/4$ and $z = 2 + i * 0.1$, $i = 0, ..., 30$. The comparisons show very good agreement between the solutions computed from Coustyx and analytical expressions.

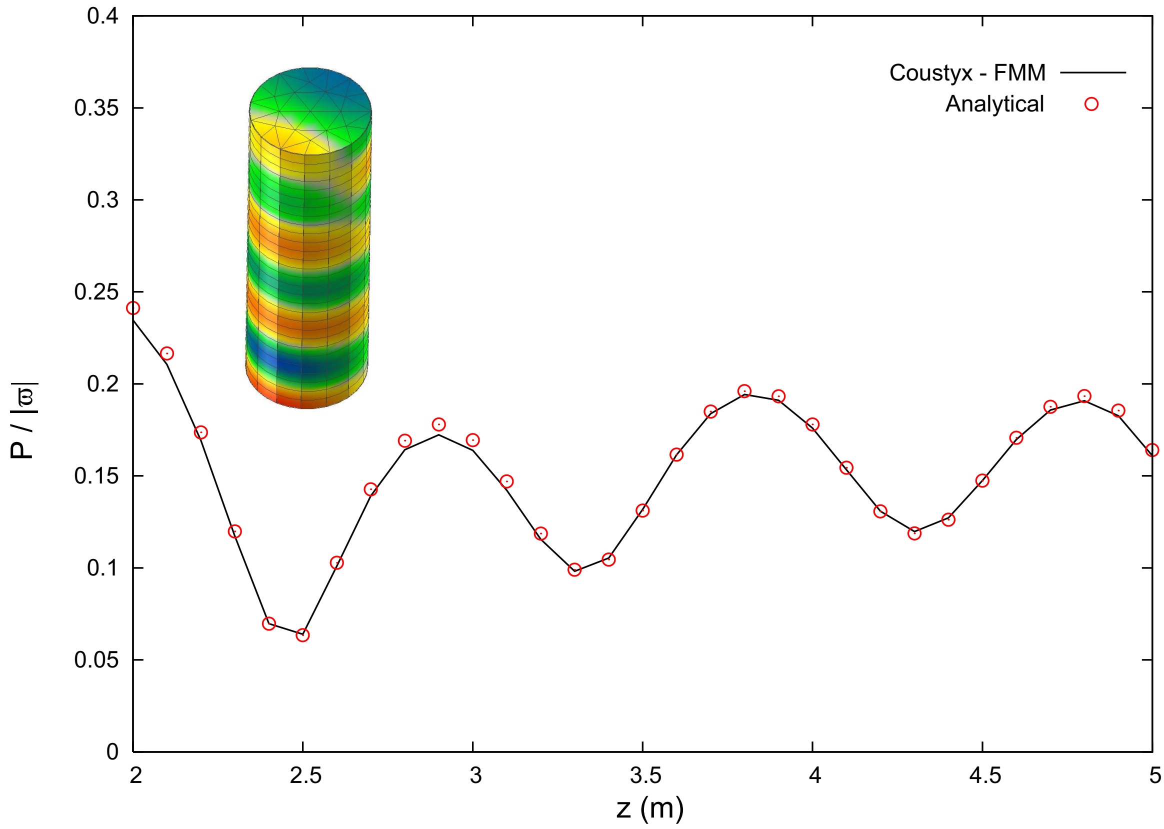

Figure 2: Field point pressure comparisons inside a rigid cylinder with a point source from Coustyx and analytical methods. Note that P is the field point pressure and $\varpi = ik\rho_o c\beta_o$, where β_o is the volume velocity of the monopole source.

References

[1] M. Abramowitz and I. E. Stegun. *Handbook of Mathematical Functions, AMS55.* U.S. Department of Commerce, National Bureau of Standards, 1972.