Scattering of a Plane Wave by Two Concentric Penetrable Spheres

1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model the scattering of a plane wave by two concentric penetrable spheres immersed in an infinite domain.
- Derive an analytical solution.
- Validate Coustyx software by comparing Coustyx results to the analytical solution.

2 Model description

This model is used to study scattering of a plane wave by two concentric penetrable spheres immersed in an infinite domain **Outer**. The radius of the inner sphere is $a_1 = 1\,\mathrm{m}$. The fluid medium inside the inner sphere is **InnerFluid** with an ambient density $\rho_1 = 1000\,\mathrm{kg/m^3}$ and sound speed $c_1 = 1500\,\mathrm{m/s}$. The outer sphere has a radius of $a_2 = 1.25\,\mathrm{m}$. The fluid medium in the annular region betwen the two spheres is **BetweenFluid** and has an ambient density $\rho_2 = 800\,\mathrm{kg/m^3}$ and sound speed $c_2 = 1324\,\mathrm{m/s}$. The fluid medium in the region exterior to the outer sphere is **OuterFluid** with an ambient density $\rho_3 = 791\,\mathrm{kg/m^3}$ and sound speed $c_3 = 1121\,\mathrm{m/s}$. The boundaries between at regions are perfectly transparent, meaning that the sound pressure and the normal component of velocity are continuous across the fluid boundaries.

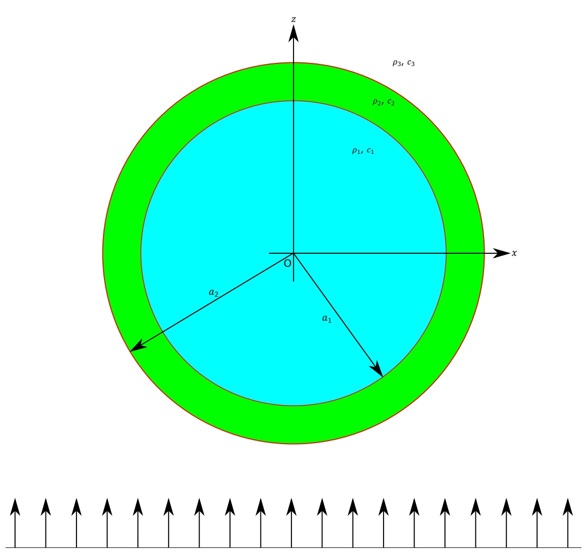

Figure 1: Schematic of the problem: Two concentric spheres with different fluid media immersed in an infinite medium and excited by a plane wave.

A **MultiDomain** model is built in Coustyx for this problem, as there are multiple regions each with a different fluid medium. There are two BE Meshes **InnerSphere** and **OuterSphere**. The mesh normals for both these meshes are pointing radially outward. The domain **Inner** is defined as a bounded region on the negative side of mesh **InnerSphere** with the fluid **InnerFluid**. Domain **Between** is defined as a bounded region on the positive side of mesh **InnerSphere** and negative side of mesh **OuterSphere** with the fluid **BetweenFluid**. Domain **Outer** is defined as an unbounded region on the positive side of mesh **OuterSphere** with the fluid **OuterFluid**. The **Outer** domain contains a plane wave acoustic source to model the incident pressure wave traveling in the z-direction. The frequency of the incident wave is 800 Hz.

2.1 Boundary Conditions

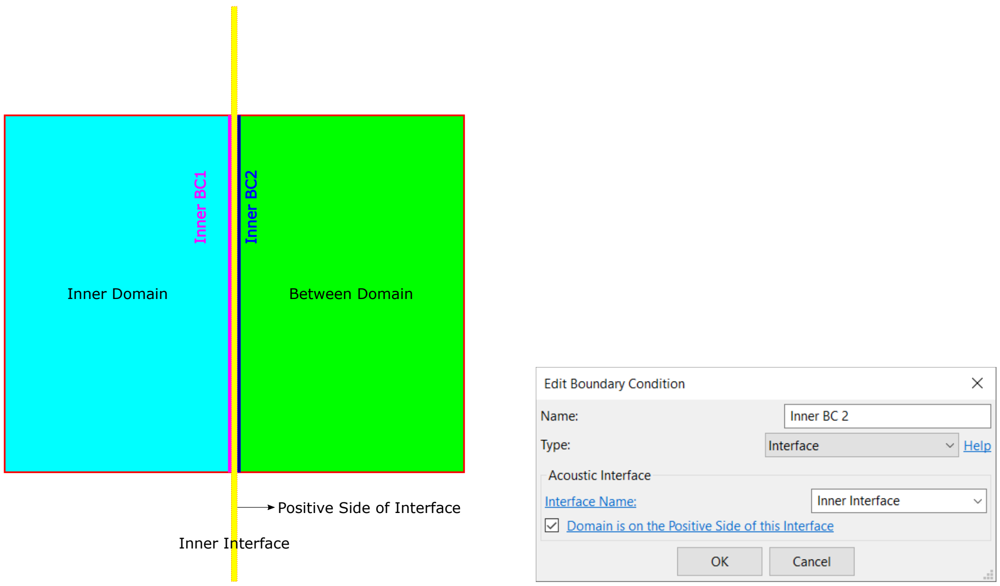

Figure 2: Modeling the inner interface, the two boundary conditions associated it and applying them to the appropriate domain.

Two transparent interfaces named **Inner Interface** and **Outer Interface** are created. **Inner Interface** separates the **Inner** domain and the **Between** domain, with the **Between** domain on the positive side of the interface, which means the interface normal goes from **Inner** domain to the **Outer** domain, as shown in Figure 2.

Two boundary conditions **Inner BC 1** and **Inner BC 2** associated with **Inner Interface** are created. **Inner BC 2** is applied to the domain is on positive side of **Inner Interface**. Hence **Inner BC 2** is applied the to the elements of **InnerSphere** when it is considered as part of **Between** domain. **Inner BC 1** boundary condition is applied to the elements of **InnerSphere** when it is considered as part of the **Inner** domain. Similarly, boundary conditions **Outer BC 1** and **Outer BC 2** are applied to the BE Mesh **OuterSphere** when it is considered as part of **Between** and **Outer** domains respectively.

3 Analytical solution

The incident pressure field in **Outer** domain is given as

$$p_i = P_o \exp(ik_3 z) \tag{1}$$

Since the pressure variation is only along the z-direction, and is axisymmetric due to the boundary conditions and geometry, the following forms can be assumed for the pressure field in each of the regions.

$$p_1(r,\theta,\phi) = \sum_{l=0}^{L} A_l j_l(k_1 r) P_l(\cos\theta) \tag{2}$$

In Equation 2, $p_1(r,\theta,\phi)$ is the pressure in Region 1, j_l represents the spherical Bessel function of order l and P_l is the Legendre polynomial of degree l. As the domain is finite and the pressure field is bounded at the origin, spherical Hankel functions are not used in the expansion in Equation 2.

$$p_2(r,\theta,\phi) = \sum_{l=0}^{L} [C_l j_l(k_2 r) + D_l h_l(k_2 r)] P_l(\cos\theta) \tag{3}$$

In Equation 3, $p_2(r,\theta,\phi)$ is the pressure in Region 2, j_l, h_l represent spherical Bessel function and spherical Hankel function of order l, and P_l is the Legendre polynomial of degree l. Both spherical Bessel and Hankel functions are used in the expansion as the **Between** domain is bounded and does not include the origin (Spherical Hankel functions have a singularity at the origin).

$$p_{3s}(r,\theta,\phi) = \sum_{l=0}^{L} F_l h_l(k_3 r) P_l(\cos\theta) \tag{4}$$

$$p_{3t}(r,\theta,\phi) = p_{3s}(r,\theta,\phi) + p_i \tag{5}$$

The total pressure in Region 3 is the sum of the scattered field given by p_{3s} and the incident field given by p_i. The scattered field only has spherical Hankel function terms in order to satisfy the Sommerfeld's radiation condition (only outgoing waves) at infinity.

The radial velocity v_r is related to the gradient of pressure.

$$v_r = \frac{1}{ikZ}\frac{\partial p}{\partial r} = \frac{1}{ik\rho c}\frac{\partial p}{\partial r} \tag{6}$$

In Equation 6, $k = \omega/c$ is the wavenumber and $Z = \rho c$ is the characteristic impedance of the fluid medium. Taking Equation 2 as an example, the radial velocity in Region 1 v_{1r} is expressed as

$$\begin{aligned} v_{1r}(r,\theta,\phi) &= \frac{1}{ik_1 Z_1} \sum_{l=0}^{L} A_l \frac{\partial}{\partial r} j_l(k_1 r) P_l(\cos\theta) \\ &= \frac{1}{iZ_1} \sum_{l=0}^{L} A_l \frac{\partial}{\partial (k_1 r)} j_l(k_1 r) P_l(\cos\theta) \\ &= \frac{1}{iZ_1} \sum_{l=0}^{L} A_l j_l'(k_1 r) P_l(\cos\theta) \end{aligned} \tag{7}$$

where j_l' is the derivative of the spherical Bessel function (with respect to its argument). To compute the derivative of the spherical Bessel and Hankel functions, the following recurrence relation is useful.

$$\frac{\partial}{\partial z} j_n(z) = \frac{n}{z} j_n(z) - j_{n+1}(z) \tag{8}$$

The incident plane wave p_i can be decomposed as series of spherical Bessel functions as follows.

$$p_i = P_o \exp(ik_3 z) = P_o \sum_{l=0}^{L} i^l j_l(k_3 r) P_l(\cos\theta) \tag{9}$$

Now we have all the necessary equations to set up the pressure and radial velocity continuity equations at the boundaries given by $r = a_1$ and $r = a_2$.

$$\begin{bmatrix} 0 & j_l(k_2a_2) & h_l(k_2a_2) & -h_l(k_3a_2) \\ 0 & j_l'(k_2a_2)/Z_2 & h_l'(k_2a_2)/Z_2 & -h_l'(k_3a_2)/Z_3 \\ j_l(k_1a_1) & -j_l(k_2a_1) & -h_l(k_2a_1) & 0 \\ j_l'(k_1a_1)/Z_1 & -j_l'(k_2a_1)/Z_2 & -h_l'(k_2a_1)/Z_2 & 0 \end{bmatrix} \left\{ \begin{array}{c} A_l \\ C_l \\ D_l \\ F_l \end{array} \right\} = \left\{ \begin{array}{c} P_o(2l+1)i^l j_l(k_3a_2) \\ P_o(2l+1)i^l j_l'(k_3a_2)/Z_3 \\ 0 \\ 0 \end{array} \right\} \quad (10)$$

In Equation 10, the first equation is the pressure continuity at $r = a_2$, the second equation is the radial velocity continuity at a_2 while the third and fourth equations enforce the pressure and radial velocity continuity at $r = a_1$. The unknown coefficients for each order l are obtained by solving the system of equations Equation 10. We have defined several script functions in the **Context Script** of the Coustyx model to compute these coefficients. Once these coefficients are determined, Equations 2, 3 and 5 are used to compute the pressure field in each of the domains.

4 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 800$ Hz using the Fast Multipole Method (FMM) by running "`Run Validation - FMM`". In this analysis sequence, the surface results for the three domains are written out to file `output_validation_fmm.txt` in folder `coustyxtmp`. The difference between the Coustyx solution and the analytical series solution is less than 3%.

Figure 3 shows angular distribution of sound pressure amplitudes at field points in different domains. The solutions from Coustyx are compared with the analtical series solution with maximum order $L = 45$. The quantities are plotted versus the polar angle θ. $\theta = 180^o$ corresponds to the south pole (illuminated region) and $\theta = 0^o$ corresponds to north pole (shadow region) of the sphere with respect to the incident plane wave. The comparisons between the solutions computed using Coustyx and analytical expressions show excellent agreement for field points in all domains.

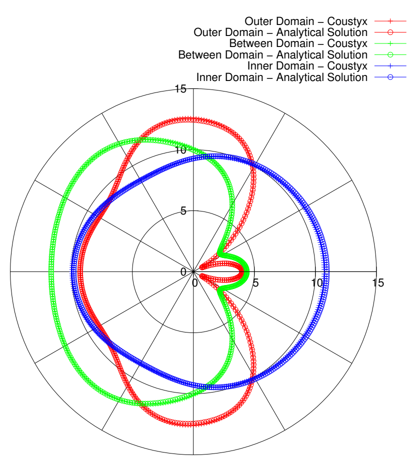

Figure 3: Total sound pressure at field points in various domains at $r = 0.5, 1.125, 1.5$ m due to incident plane wave excitation.