Two Concentric Penetrable Spheres with a Central Point Source

1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model the propagation of a spherical wave by two concentric penetrable spheres immersed in an infinite domain.
- Derive an analytical solution.
- Validate Coustyx software by comparing Coustyx results to the analytical solution.

2 Model Description

This model is used to study the propagation of a spherical wave through two concentric penetrable spheres immersed in an infinite domain **Outer**. The radius of the inner sphere is $a_1 = 1\,\mathrm{m}$. The fluid medium inside the inner sphere is **InnerFluid** with an ambient density $\rho_1 = 1000\,\mathrm{kg/m^3}$ and sound speed $c_1 = 1500\,\mathrm{m/s}$. The outer sphere has a radius of $a_2 = 1.25\,\mathrm{m}$. The fluid medium in the annular region betwen the two spheres is **BetweenFluid** and has an ambient density $\rho_2 = 800\,\mathrm{kg/m^3}$ and sound speed $c_2 = 1324\,\mathrm{m/s}$. The fluid medium in the region exterior to the outer sphere is **OuterFluid** with an ambient density $\rho_3 = 791\,\mathrm{kg/m^3}$ and sound speed $c_3 = 1121\,\mathrm{m/s}$. The boundaries between at regions are perfectly transparent, meaning that the sound pressure and the normal component of velocity are continuous across the fluid boundaries.

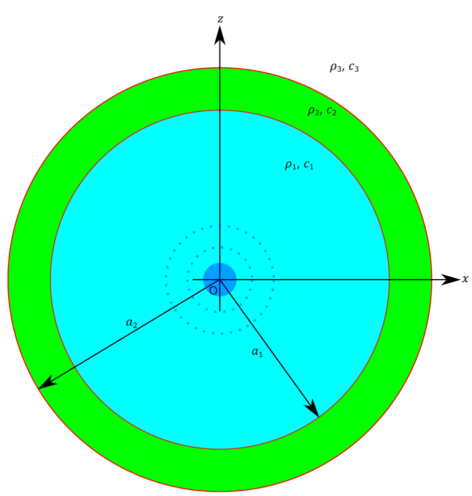

Figure 1: Schematic of the problem: Two concentric spheres with different fluid media immersed in an infinite medium and excited by a monopole source at the center.

A **MultiDomain** model is built in Coustyx for this problem, as there are multiple regions each with a different fluid medium. There are two BE Meshes **InnerSphere** and **OuterSphere**. The mesh normals for both these meshes are pointing radially outward. The domain **Inner** is defined as a bounded region on the negative side of mesh **InnerSphere** with the fluid **InnerFluid**. Domain **Between** is defined as a bounded region on the positive side of mesh **InnerSphere** and negative side of mesh **OuterSphere** with the fluid **BetweenFluid**. Domain **Outer** is defined as an unbounded region on the positive side of mesh **OuterSphere** with the fluid **OuterFluid**. The **Inner** domain contains a monopole acoustic source of source amplitude $A_s = -1$ which provides the excitation. The frequency of the incident wave is 800 Hz.

2.1 Boundary Conditions

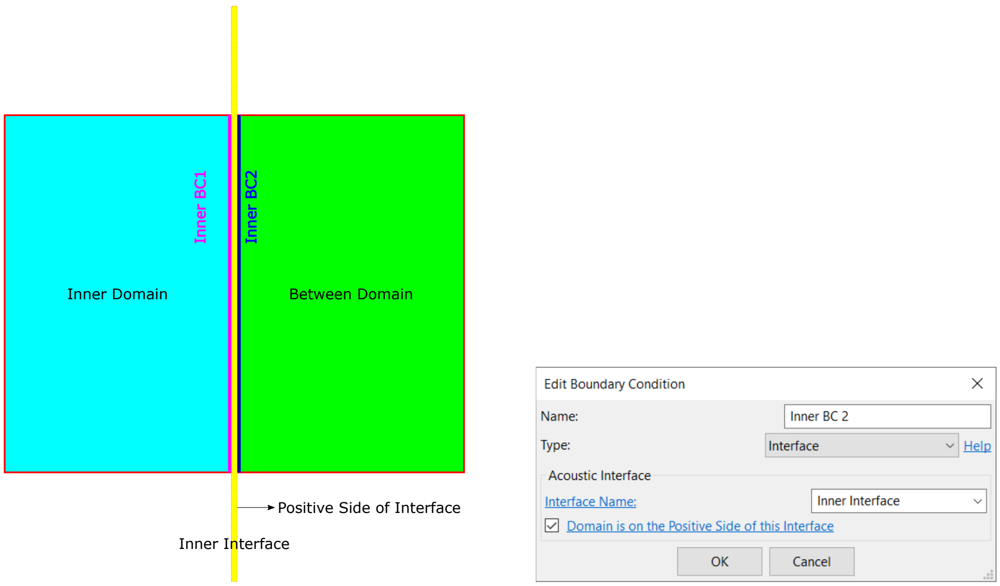

Figure 2: Modeling the inner interface, the two boundary conditions associated it and applying them to the appropriate domain.

Two transparent interfaces named **Inner Interface** and **Outer Interface** are created. **Inner Interface** separates the **Inner** domain and the **Between** domain, with the **Between** domain on the positive side of the interface, which means the interface normal goes from **Inner** domain to the **Outer** domain, as shown in Figure 2.

Two boundary conditions **Inner BC 1** and **Inner BC 2** associated with **Inner Interface** are created. **Inner BC 2** is applied to the domain is on positive side of **Inner Interface**. Hence **Inner BC 2** is applied the to the elements of **InnerSphere** when it is considered as part of **Between** domain. **Inner BC 1** boundary condition is applied to the elements of **InnerSphere** when it is considered as part of the **Inner** domain. Similarly, boundary conditions **Outer BC 1** and **Outer BC 2** are applied to the BE Mesh **OuterSphere** when it is considered as part of **Between** and **Outer** domains respectively.

3 Analytical Solution

In this problem the geometry as well as the excitation are symmetric. Hence the pressure field varies only with the distance r from the origin. If all the fluid media were identical, we would have a outgoing spherical that emanate from the source and travel outward to infinity. However, in this problem, the fluid medium in each region is different. Due to the impedance mismatch at the interfaces, in the **Inner** and **Between** domains, we will have both incoming and outgoing waves.

In the **Outer** domain, only outgoing wave will be present to satisfy the Sommerfeld's radiation condition at infinity. Thus we assume a pressure field in each of the domains as follows:

$$p_1(r,\theta,\phi) = A\frac{\exp(ik_1r)}{4\pi r} + B\frac{\exp(-ik_1r)}{4\pi r} \tag{1}$$

$$p_2(r,\theta,\phi) = C\frac{\exp(ik_2r)}{4\pi r} + D\frac{\exp(-ik_2r)}{4\pi r} \tag{2}$$

$$p_3(r,\theta,\phi) = E\frac{\exp(ik_3r)}{4\pi r} \tag{3}$$

The radial velocity v_r is related to the gradient of pressure.

$$v_r = \frac{1}{ik\rho c}\frac{\partial p}{\partial r} = \frac{1}{ikZ}\frac{\partial p}{\partial r} \tag{4}$$

In Equation 4, $k = \omega/c$ is the wavenumber and $Z = \rho c$ is the characteristic impedance of the fluid medium. Taking Equation 1 as an example, the radial velocity in Region 1 v_{1r} is expressed as

$$\begin{aligned} v_{1r}(r,\theta,\phi) &= \frac{1}{ik_1Z_1}\left\{A\frac{\exp(ik_1r)}{4\pi}\left[\frac{ik_1}{r} - \frac{1}{r^2}\right] + B\frac{\exp(-ik_1r)}{4\pi}\left[\frac{-ik_1}{r} - \frac{1}{r^2}\right]\right\} \\ &= \frac{1}{ik_1Z_1(4\pi r^2)}\left\{A\exp(ik_1r)(ik_1r-1) + B\exp(-ik_1r)(-ik_1r-1)\right\} \\ &= \frac{1}{ik_1Z_1(4\pi r^2)}\left\{A\exp(ik_1r)(ik_1r-1) - B\exp(-ik_1r)(ik_1r+1)\right\} \end{aligned} \tag{5}$$

3.1 Conditions for Determining the Unknown Coefficients

3.1.1 Source at the Origin

$$\nabla^2 p_1 + k_1^2 p_1 = A_s\delta(r) \tag{6}$$

Integrating Equation 6 over a small sphere of radius ϵ as $\epsilon \to 0$ will help us connect the source terms with the amplitudes of the outgoing wave A and incoming wave B.

$$\int_v \nabla^2 p_1 + k_1^2 p_1 \, dv = A_s \tag{7}$$

The second term on the left hand side goes to zero as p_1 varies as $1/r$.

$$\begin{aligned} \int_v \nabla^2 p_1 \, dv &= A_s \\ \int_s \frac{\partial p_1}{\partial r}\, ds &= A_s \\ -(A+B) &= A_s \\ A+B &= -A_s \end{aligned} \tag{8}$$

3.1.2 Continuity Conditions at $r = a_1$

At a point on $r = a_1$, the pressure and radial velocity when the point is considered as part of **Inner** domain or **Between** domain must be identical.

3.1.3 Continuity Conditions at $r = a_2$

At a point on $r = a_2$, the pressure and radial velocity when the point is considered as part of **Between** domain or **Outer** domain must be identical.

The source and continuity conditions yield all the necessary equations required to determine the unknown coefficients.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ e^{ik_1a_1} & e^{-ik_1a_1} & -e^{ik_2a_1} & -e^{-ik_2a_1} & 0 \\ \frac{e^{ik_1a_1}(ik_1a_1-1)}{k_1Z_1} & \frac{-e^{-ik_1a_1}(ik_1a_1+1)}{k_1Z_1} & \frac{-e^{ik_2a_1}(ik_2a_1-1)}{k_2Z_2} & \frac{e^{-ik_2a_1}(ik_2a_1+1)}{k_2Z_2} & 0 \\ 0 & 0 & e^{ik_2a_2} & e^{-ik_2a_2} & -e^{ik_3a_2} \\ 0 & 0 & \frac{e^{ik_2a_2}(ik_2a_2-1)}{k_2Z_2} & \frac{-e^{-ik_2a_2}(ik_2a_2+1)}{k_2Z_2} & \frac{-e^{ik_3a_2}(ik_3a_2-1)}{k_3Z_3} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \\ E \end{Bmatrix} = \begin{Bmatrix} -A_s \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{9}$$

In Equation 9, the first equation is source condition at the origin, the second and third equations are pressure and radial velocity continuity at a_1 while the fourth and fifth equations enforce the pressure and radial velocity continuity at $r = a_2$. We have defined script functions in the **Context Script** of the Coustyx model to compute these coefficients. Once these coefficients are determined, Equations 1, 2 and 3 are used to compute the pressure field in each of the domains.

4 Results and Validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 800\,\text{Hz}$ using the Fast Multipole Method (FMM) by running "`Run Validation - FMM`". In this analysis sequence, the surface results for the three domains are written out to file `output_validation_fmm.txt` in folder `coustyxtmp`. The difference between the Coustyx solution and the analytical solution is less than 1.21%. The sound pressure along a radial line in the z-direction is also computed using Coustyx and the analytical solution. The results are written to file `coustyxtmp/sensor_comparison_fmm.txt`. Both solutions are within 0.6% of each other. Figure 3 shows the variation of sound pressure with the distance from the center, where the solutions from Coustyx are compared with the analytical solution; excellent agreement is observed. As expected, for a point monopole source, the overall pressure varies inversely with distance, however the effects of dissimilar media are clearly seen at the interface boundaries as a change in slope.

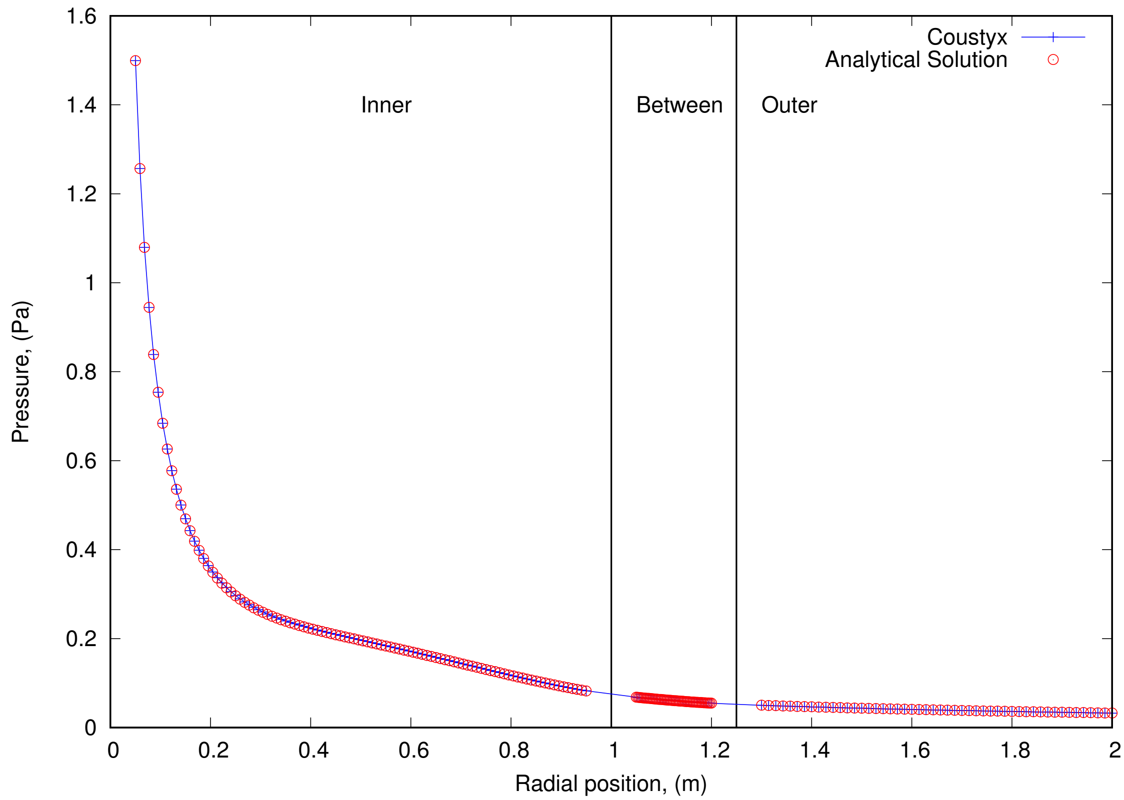

Figure 3: Sound pressure at field points in various domains at due to excitation by a point monopole source at the center.