Circular Piston in a Plane Baffle

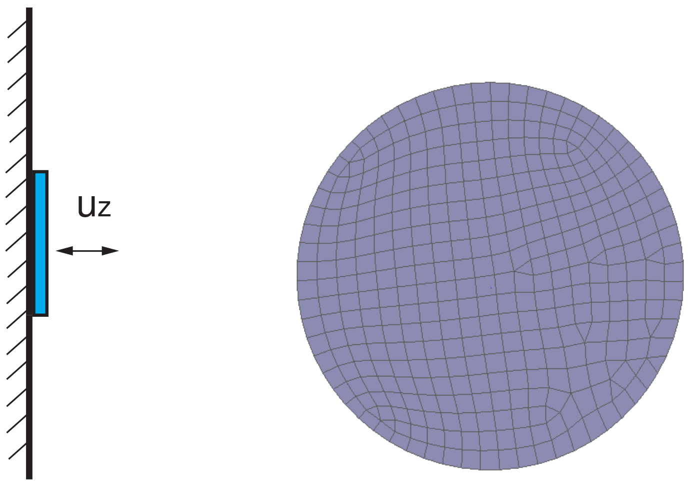
1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model a circular piston in an infinite rigid plane baffle using a MultiDomain model.
- Derive far-field analytical solution for the baffled circular piston problem [1].
- Validate Coustyx software by comparing Coustyx results to analytical solutions.

2 Model description

We model a circular piston of radius $a = 1\,\text{m}$. We place the piston on an infinite rigid plane baffle. The baffle plane divides the space into two semi-unbounded domains. We are interested in the field on the radiating side of the piston. The fluid medium is assumed to be air with sound speed $c_o = 343\,\text{m/s}$ and mean density $\rho_o = 1.21\,\text{kg/m}^3$. The characteristic impedance of air $Z_o = \rho_o c_o = 415.03\,Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c_o$. The piston is assumed to be vibrating with a uniform velocity $u_z = 1\,\text{m/s}$ and the baffle plane is assumed to be perfectly reflecting. The BE mesh of the piston is shown in Figure 1.



(a) (b)

Figure 1: Acoustic problem description. (a) Circular piston in a rigid baffle, and (b) Boundary element mesh

3 Boundary Conditions

The piston is vibrating with a uniform normal velocity $u_z = 1$. In Coustyx, this boundary condition is applied as an "Uniform Normal Velocity", $v_n = -u_z$; where v_n is the velocity in the direction of *Domain Normal*. Note that all boundary conditions in a MultiDomain model are defined with respect to the *Domain Normal*, which always points away from the domain of interest. The baffle plane is assumed to be perfectly reflecting and hence the acoustic normal velocity on it is equal to zero. In Coustyx a baffle plane is defined by a point on the plane and its normal using the feature "Planes".

4 Analytical solution

The pressure field due to a vibrating piston placed in an infinite rigid baffle is given by *Rayleigh integral* [1],

$$p(P) = -2ikZ_0 \int_S u_z(Q) G_\infty(P,Q) dS(Q) \tag{1}$$

where P is the observation point, Q is a point on the piston, $u_z(Q)$ is the velocity at Q, $G_\infty(P,Q) = \frac{e^{ikR}}{4\pi R}$ is the free-space Green's function where R is the distance between P and Q, and the integration is over the surface area S of the piston.

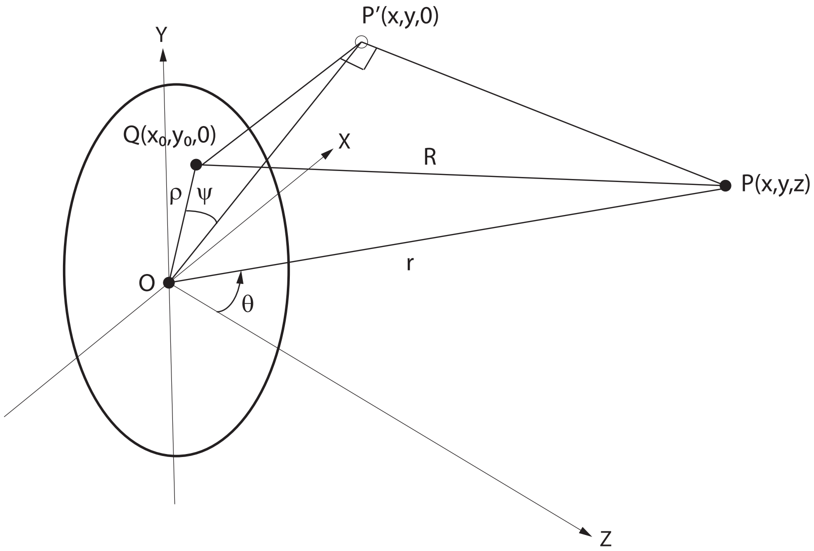


Figure 2: Definition of variables.

The definition of various variables are shown in Figure 2. Using the triangle cosine rule and the pythagoras theorem, the distance R between the observation point P and source point Q can be written as,

$$R^2 = r^2 + \rho^2 - 2\rho r \sin\theta \cos\psi$$

where r is the distance between the center of the piston O to the observation point P, ρ is the radial distance of point Q from O, $\theta = \frac{z}{r}$, and ψ is the angle between $\overline{OQ}$ and $\overline{OP'}$, where P' is the projection of P on to the baffle plane.

Far-field approximation, $r \gg a$, and $r \gg \rho$, gives

$$R \approx r - \rho \sin\theta \cos\psi$$
$$\frac{1}{R} \approx \frac{1}{r}$$

Therefore, the far-field pressure radiated by a baffled circular piston vibrating with constant velocity u_z is given by

$$\begin{aligned} p(P) &= -ikZ_0 u_z \tfrac{e^{ikr}}{2\pi r} \int_S e^{-ik\rho \sin\theta \cos\psi} dS(Q) \\ &= -ikZ_0 u_z \tfrac{e^{ikr}}{2\pi r} \int_0^a \left[\rho d\rho \int_0^{2\pi} e^{-ik\rho \sin\theta \cos\psi} d\psi \right] \end{aligned} \tag{2}$$

We use the following properties of 0^{th} and 1^{st} order Bessel functions,

$$\begin{aligned} J_0(\alpha) &= \tfrac{1}{2\pi} \int_0^{2\pi} e^{-i\alpha \cos\psi} d\psi \\ J_1(\beta) &= \tfrac{1}{\beta} \int_0^{\beta} \alpha J_0(\alpha) d\alpha \end{aligned} \tag{3}$$

Finally, the far-field pressure radiated by a baffled circular piston is given by

$$p(P) = -ikZ_0 u_z \frac{e^{ikr}}{2\pi r} \pi a^2 \left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta} \right] \tag{4}$$

Note that for $\theta = 0$,

$$\frac{2J_1(ka\sin\theta)}{ka\sin\theta} = 1$$

5 Results and Validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 327.54Hz$ (that is, $ka = 6$) using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".

Coustyx uses Direct BE method to solve the radiation problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Field point solutions are then computed from the surface solutions.

Figure 3 shows angular distribution of the field point pressure amplitude at $r = 100m$ computed from both Coustyx and analytical methods for different values of ka. The field points are assumed to lie on $x = 0$ plane. The quantities are plotted versus the polar angle θ. The comparisons show very good agreement between Coustyx and analytical expressions. Note that when the wave length of the sound is larger than the radius of the piston, that is $ka \ll 1$, the sound spreads out uniformly in all directions from the piston as seen for the case $ka = 0.5$. However, at high frequencies (smaller wavelength), that is $ka \gg 1$ the radiated sound shows significant directivity with sound chiefly radiated as a beam perpendicular to the baffle as evident for the cases $ka = 3$, and $ka = 6$.

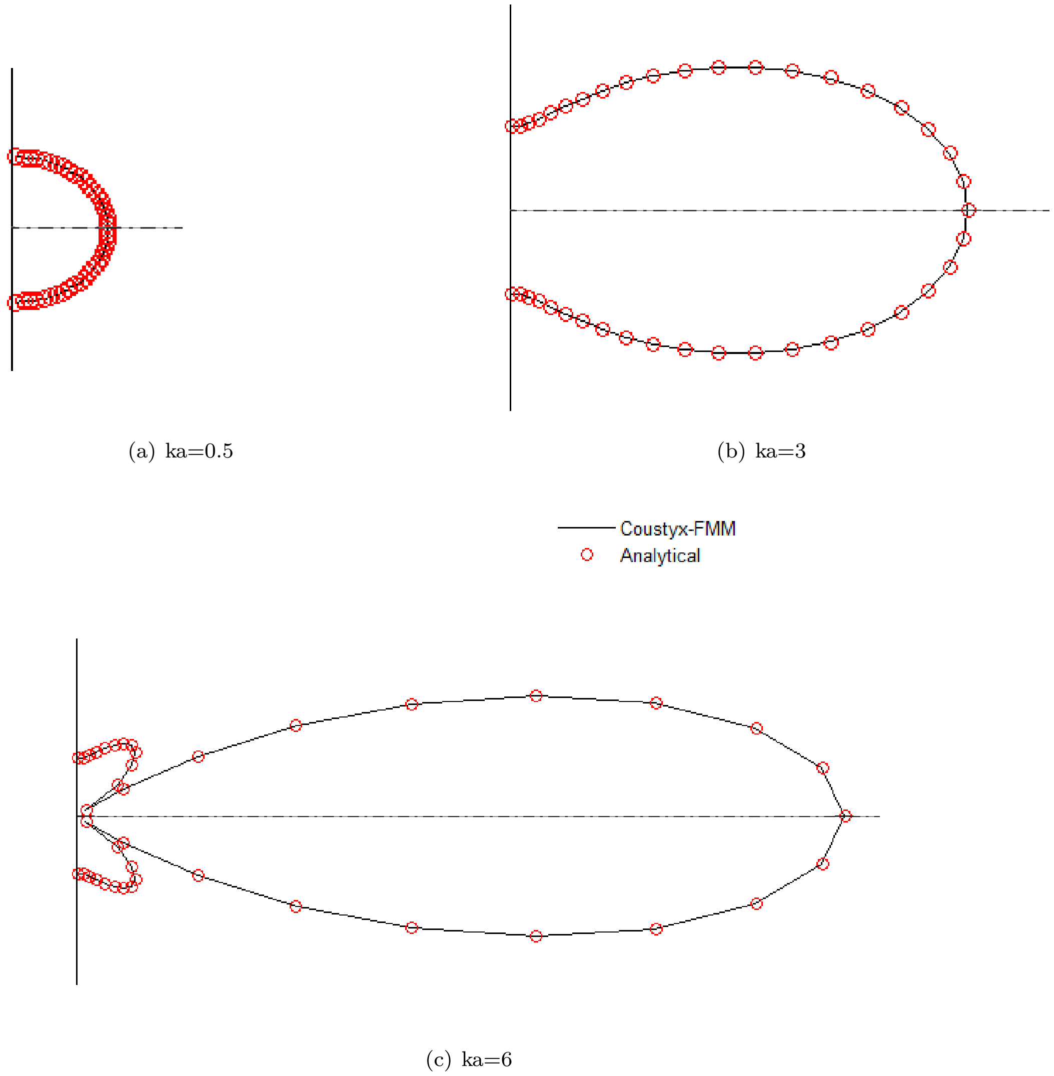


Figure 3: Angular distribution of field point pressure amplitudes on the plane $x = 0$ at $r = 100m$ due to a vibrating piston placed in an infinite rigid baffle from Coustyx and analytical methods. The quantities are plotted versus the polar angle θ for different values of ka.

References

[1] Philip M. Morse and K. Uno Ingard. *Theoretical Acoustics*, pages 381–382. Princeton university Press, Princeton, New Jersey, 1986.