Radiation from a Hemisphere in a Rigid Baffle

1 Introduction

The main objectives of this Demo Model are to

- Illustrate the sound radiation from a hemispherical dome tweeter, by considering the problem of a hemisphere mounted in an infinite rigid baffle. The hemisphere is vibrating as a rigid body along its axis (z-axis) with an amplitude v_a.
- This problem is amenable to an analytical solution using spherical harmonic expansion. The numerical solution from Coustyx is compared with the analytical solution to verify for solution accuracy. Note that the analytical solution is a series expansion and sufficient terms have to included to get an accurate value. Here, 120 terms are considered (only the even orders have a non-zero contribution).
- We are interested in the characteristics of the sound field in the fluid medium surrounding the hemisphere - such as variation of sound pressure with distance, directivity, radiated sound power, and efficiency.
- In this example, we use the HIE collocation method for the BEM solution. Note that the HIE formulation is susceptible to the irregular frequency issue, which can be remedied using CHIEF points. In a later example, we will use the Burton Miller collocation that uses a combination of HIE and the normal derivative integral equation and yields accurate solution at all frequencies without needing to use CHIEF points.

2 Model description

A hemisphere of radius $a = 40\,\text{mm}$, with its center at the origin and z-axis as the axis of symmetry is set in an infinite rigid baffle (xy-plane). The radius of the hemisphere is chosen to be relatively small, representative of a typical dome tweeter. The BE mesh of the hemisphere is shown in Figure 1.

We choose the following system of units for the Coustyx model: Length is in millimeter (mm), Force is in Newton (N) and time is in seconds (s). Choose the `millimeter - newton - second` option in the Edit Units dialog box. All other derived quantities are in terms of these primary units. For example, sound pressure will be in N/mm^2 or MPa, Sound Intensity in $\text{N/(mm} \cdot \text{s)}$ etc.

The fluid medium surrounding the sphere is air with sound speed $c = 343\,\text{m/s} = 343,000\,\text{mm/s}$ and ambient density $\rho_0 = 1.21\,\text{kg/m}^3 = 1.21\,\text{Ns}^2/\text{m}^4 = 1.21e - 12\,\text{Ns}^2/\text{mm}^4$. The characteristic impedance of air $Z_0 = \rho_0 c = 4.1503e - 7\,\text{Ns/mm}^3$. The wavenumber at a frequency f is given as $k = \omega/c = 2\pi f/c$ and has the units of 1/mm. As the hemisphere is small (radius = 40 mm), even at high frequency the value of ka will be small. For example, At 20 KHz, $ka = 2\pi f/c \cdot a = 14.65$, and $ka = 1$ around 1365 Hz. At frequencies higher than 1365 Hz, we can expect the speaker to be a good radiator.

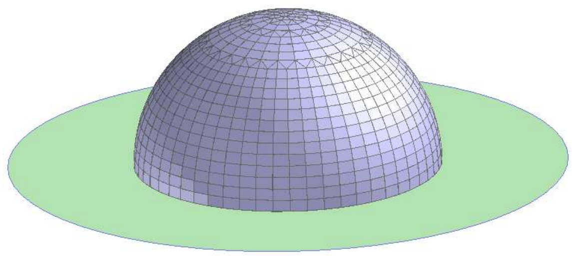

Figure 1: Boundary element mesh of a hemisphere with radius $a = 40$mm vibrating in a rigid baffle plane.

A radial velocity distribution, $\tilde{v}_r$ on the hemisphere corresponding to the vertical rigid body motion $v_z = v_a$ is given as,

$$\tilde{v}_r(\theta, \phi) = v_a \cos\theta \quad 0 \leq \theta \leq \pi/2 \tag{1}$$

In this example v_a was taken as 1 mm/s.

3 Boundary Conditions

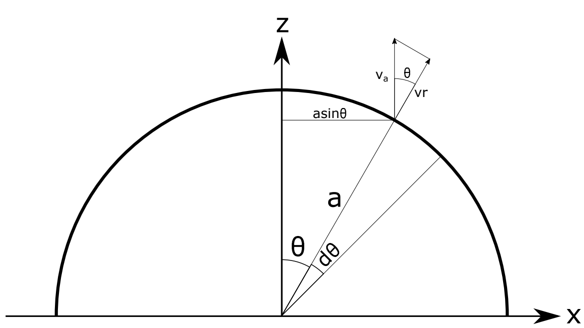

Figure 2: Schematic of the hemisphere in the baffle.

The sphere is vibrating with a radial velocity $\tilde{v}_r$ given by Equation 1. In Coustyx, this boundary condition is applied as an "Nonuniform Normal Velocity" and is defined by script. The normal velocity, $v_n = -\tilde{v}_r$, is the velocity in the direction of *Domain Normal*. Note that all boundary conditions in a MultiDomain model are defined with respect to the *Domain Normal*, which always points **away** from the domain of interest. For this example, as the exterior domain is our domain of interest; the domain normal is pointing away from the exterior domain, into the sphere.

Another way to specify the velocity boundary condition is to use the `Uniform Velocity` option to specify a uniform vector velocity. In this example, this is given given by the boundary condition named `Vector Velocity`.

Both ways of creating the boundary condition are equivalent. The user must remember to **apply** the boundary condition to the mesh elements. If a boundary condition is not applied to the elements, they will have a default zero normal velocity boundary condition corresponding to a rigid boundary.

4 CHIEF points

In Direct BEM radiation problems, CHIEF (Combined Helmholtz Integration Equation Formulation) points are used to eliminate large errors in the solution at certain frequencies which are the natural frequencies of the complementary interior problem. At these frequencies the Helmholtz equation doesn't yield a unique solution. We define a few arbitrary CHIEF points inside the sphere to additionally constrain the problem to obtain accurate solutions. Please note that when a CHIEF point falls on an interior nodal surface it does not provide additional constraint effect at that frequency. Hence, selection of good CHIEF points is crucial in obtaining accurate solutions at all frequencies. One way to ensure this is to use the `Auto Generate` option in Coustyx to create CHIEF points at random interior locations.

5 Analytical solution

5.1 Sound Pressure:

The first step in the solution procedure is to expand the applied radial velocity as a series of Legendre polynomials $P_m(\cos\theta)$. If we consider the radial velocity distribution over the full sphere, taking into account the symmetry caused by the rigid baffle plane, it will be as follows:

$$v_r(a,\theta,\phi) = \begin{cases} v_a\cos\theta & 0 \le \theta \le \pi/2 \\ -v_a\cos\theta & \pi/2 < \theta \le \pi \end{cases} \tag{2}$$

As this is an even function over $\theta = \pi/2$, only the Legendre polynomials of even degree contribute to the expansion.

$$v_r(a,\theta,\phi) = \sum_{m=0}^{\infty} V_m P_m(\cos\theta) \tag{3}$$

The expression for the expansion coefficients V_m is found using the orthogonality properties of the Legendre polynomials.

$$V_m = \begin{cases} 0,\ m \text{ odd} \\ \frac{v_a}{2},\ m = 0 \\ -v_a\left[\frac{(m+1)}{(2m+3)}P_{m+2}(0) + \frac{(2m+1)}{(2m-1)(2m+3)}P_m(0) - \frac{m}{(2m-1)}P_{m-2}(0)\right],\ m \ne 0 \text{ even} \end{cases} \tag{4}$$

The next step is to assume a solution as a series of elementary solutions each of which satisfies the wave equation exactly.

$$p(r,\theta,\phi) = \sum_{m=0}^{\infty} A_m h_m(kr) P_m(\cos\theta) \tag{5}$$

where h_m are spherical Hankel functions of the first kind of order m. The unknown coefficients A_m are determined by equating the surface radial velocity from Equation 5 to the applied radial velocity coefficients V_m.

$$\frac{\partial p(r,\theta,\phi)}{\partial r} = \sum_{m=0}^{\infty} A_m h'_m(kr) k P_m(\cos\theta) \tag{6}$$

In Equation 6 the prime in h'_m refers to the derivative of the Hankel function with respect to its *argument* (namely, kr).

$$v_r(r,\theta,\phi) = \frac{1}{ikZ_0}\frac{\partial p(r,\theta,\phi)}{\partial r} = \frac{1}{ikZ_0}\sum_{m=0}^{\infty} A_m h'_m(kr) k P_m(\cos\theta) \tag{7}$$

The coefficients A_m are determined by equating Equation 7 to the applied surface radial velocity expansion Equation 3.

$$A_m = \frac{V_m(iZ_0)}{h'_m(ka)} \tag{8}$$

Therefore, the expression for the pressure field is given as

$$p(r,\theta,\phi) = (iZ_0)\sum_{m=0}^{\infty} V_m \frac{h_m(kr)}{h'_m(ka)} P_m(\cos\theta) \tag{9}$$

Equation 9 is the primary equation for the pressure field. All the other acoustic quantities, namely components of velocity, expressions for radiated and reactive power, and radiation efficiency are derived from this.

5.2 Acoustic Particle Velocity:

The velocity field is related to the pressure gradient as follows:

$$\begin{aligned}\vec{v} &= \frac{1}{ikZ_0}\nabla p \\ &= \frac{1}{ikZ_0}\left(\frac{\partial p}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\hat{e}_\phi\right) \end{aligned} \tag{10}$$

Therefore the radial component of velocity v_r is given by

$$\begin{aligned} v_r(r,\theta,\phi) &= \frac{1}{ikZ_0}\frac{\partial p}{\partial r} \\ &= \frac{1}{ikZ_0}(iZ_0)\sum_{m=0}^{\infty} V_m \frac{h'_m(kr)k}{h'_m(ka)} P_m(\cos\theta) \\ &= \sum_{m=0}^{\infty} V_m \frac{h'_m(kr)}{h'_m(ka)} P_m(\cos\theta) \end{aligned} \tag{11}$$

If we evaluate Equation 11 at $r = a$, it reduces to Equation 3, which is an additional check the that the analytical solution as derived is correct. The polar component of velocity v_θ is given by

$$\begin{aligned} v_\theta(r,\theta,\phi) &= \frac{1}{ikZ_0}\frac{1}{r}\frac{\partial p}{\partial \theta} \\ &= \frac{1}{ikZ_0}\frac{1}{r}(iZ_0)\sum_{m=0}^{\infty} V_m \frac{h_m(kr)}{h'_m(ka)} P'_m(\cos\theta)(-\sin\theta) \\ &= \frac{-\sin\theta}{kr}\sum_{m=0}^{\infty} V_m \frac{h_m(kr)}{h'_m(ka)} P'_m(\cos\theta) \end{aligned} \tag{12}$$

The azimuthal component of velocity v_ϕ is given by Equation 13. It is zero as the problem is axisymmetric.

$$\begin{aligned} v_\phi(r,\theta,\phi) &= \frac{1}{ikZ_0}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} \\ &= 0 \end{aligned} \tag{13}$$

Coustyx has built in functions for Legendre polynomials and spherical Hankel functions of the first kind. They are invoked as follows:

```
var Legendre_P_array = LegendreP(n,x);
```

The return value `Legendre_P_array` is an array of size `n+1` containing $P_0(x), P_1(x), \ldots P_n(x)$.

Spherical Hankel functions of the first kind are obtained by using the function `h1n`.

```
var hn_kr = h1n(min_order, n_terms, kr);
```

The return value `hn_kr` is an array of size `n_terms` containing all the orders from `min_order` to `min_order+n_terms-1`.

The following recurrence relations are used to compute the derivatives of the spherical Hankel functions and Legendre polynomials. They are listed here, as some of the references have typos and yield incorrect values as a result.

$$h'_n(z) = -h_{n+1}(z) + \frac{n}{z}h_n(z) \tag{14}$$

$$P'_{m+1}(x) = P'_{m-1}(x) + (2m+1)P_m(x) \tag{15}$$

5.3 Sound Power:

The time averaged radiated sound power W_a and the reactive power W_r are obtained by integrating the active and reactive intensity on the surface of the sphere. Each term in the spherical harmonic expansion radiates independently, and the cross terms evaluate to zero because of the orthogonality property of the Legendre polynomials.

5.3.1 Radiated Sound Power W_a:

$$\begin{aligned} W_a &= \frac{1}{2}\Re \int p v_r^* dS \\ &= \frac{1}{2}\Re \int \sum_{m=0}^{\infty} (iZ_0) |V_m|^2 \frac{h_m(ka)}{h'_m(ka)} P_m^2(\cos\theta) dS \\ &= \frac{1}{2}\Re \left(iZ_0 \sum_{m=0}^{\infty} |V_m|^2 \frac{h_m(ka)}{h'_m(ka)} \int P_m^2(\cos\theta) dS \right) \\ &= \frac{1}{2}\Re \left(iZ_0 \sum_{m=0}^{\infty} |V_m|^2 \frac{h_m(ka)}{h'_m(ka)} \frac{2\pi a^2}{2m+1} \right) \\ &= \pi a^2 Z_0 \sum_{m=0}^{\infty} \frac{|V_m|^2}{2m+1} \Re \left(\frac{i h_m(ka)}{h'_m(ka)} \right) \end{aligned} \tag{16}$$

5.3.2 Reactive Sound Power W_r:

$$\begin{aligned} W_a &= \frac{1}{2}\Im \int p v_r^* dS \\ &= \frac{1}{2}\Im \int \sum_{m=0}^{\infty} (iZ_0) |V_m|^2 \frac{h_m(ka)}{h'_m(ka)} P_m^2(\cos\theta) dS \\ &= \frac{1}{2}\Im \left(iZ_0 \sum_{m=0}^{\infty} |V_m|^2 \frac{h_m(ka)}{h'_m(ka)} \int P_m^2(\cos\theta) dS \right) \\ &= \frac{1}{2}\Im \left(iZ_0 \sum_{m=0}^{\infty} |V_m|^2 \frac{h_m(ka)}{h'_m(ka)} \frac{2\pi a^2}{2m+1} \right) \\ &= \pi a^2 Z_0 \sum_{m=0}^{\infty} \frac{|V_m|^2}{2m+1} \Im \left(\frac{i h_m(ka)}{h'_m(ka)} \right) \end{aligned} \tag{17}$$

5.3.3 Input Power W_i:

$$\begin{aligned} W_i &= \frac{1}{2} Z_0 \int v_n^2 dS \\ &= \frac{1}{2} Z_0 \int_0^{\frac{\pi}{2}} v_a^2 \cos^2\theta \, 2\pi a^2 \sin\theta d\theta \\ &= \frac{\pi a^2 Z_0 v_a^2}{3} \end{aligned} \tag{18}$$

5.3.4 Radiation Efficiency σ:

$$\sigma = \frac{W_a}{W_i} \tag{19}$$

6 Results and Validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc.

Before running an analysis sequence that has multiple frequency lines, a single frequency analysis is run, as a sanity check. Once we have verified that the model runs without any issues and the results make sense, we can run an analysis sequence that contains multiple frequencies. An example of an analysis sequence with single frequency is `Run Demo`.

Here, an analysis sequence `Run Demo` is run. This runs the model at 1000 Hz. Field points are set up along radial lines as shown in going from $r = 100$ mm to $r = 2000$ mm as shown in the Figure 3. Two files `sensors_demo.dat` (Coustyx BEM) and `sensors_analytical_demo.dat` (Analytical Solution) are created during the run. The pressure and velocity data from these files are read into Python `numpy` arrays using the `read_sensor_file` function, in the module `process_coustyx__data` and plotted using the `matplotlib` library. This module also contains function `read_power_file` for reading the sound power data generated by Coustyx.

From these runs the sensor data can be imported into Python as numpy arrays as follows:

```
frequencies,pressure,vx,vy,vz = read_sensor_file(r'sensors_validation_no_fmm.dat')
```

Power data can be imported as follows:

```
frequencies, radiated_power, reactive_power, input_power, radiation_efficiency =
  read_power_file(r'power_validation_no_fmm.dat')
```

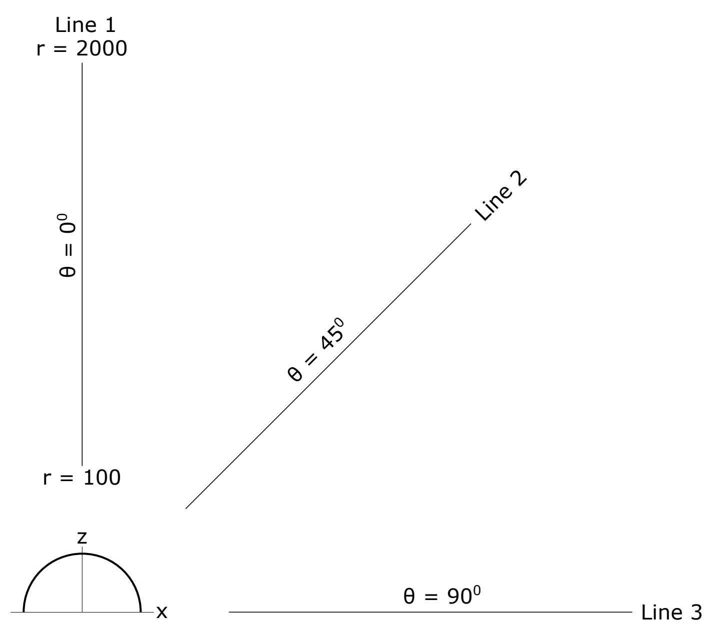

Figure 3: Location of field points relative to the hemisphere.

Figure 4 shows how the pressure varies with increasing distance along the three lines from Figure 3. We can see that there is excellent agreement between Coustyx and the analytical solution using maximum order of $m = 120$. There is very good agreement along `Line 2` and `Line 3` as well.

The maximum percentage difference in field point sound pressure between the Coustyx solution and the analytical solution was 0.27%. Each line in Figure 4 represents one frequency. The first five frequencies from 500 Hz to 2500 Hz are shown. From Figure 4, a $1/r$ type of pressure variation with distance is clearly observed.

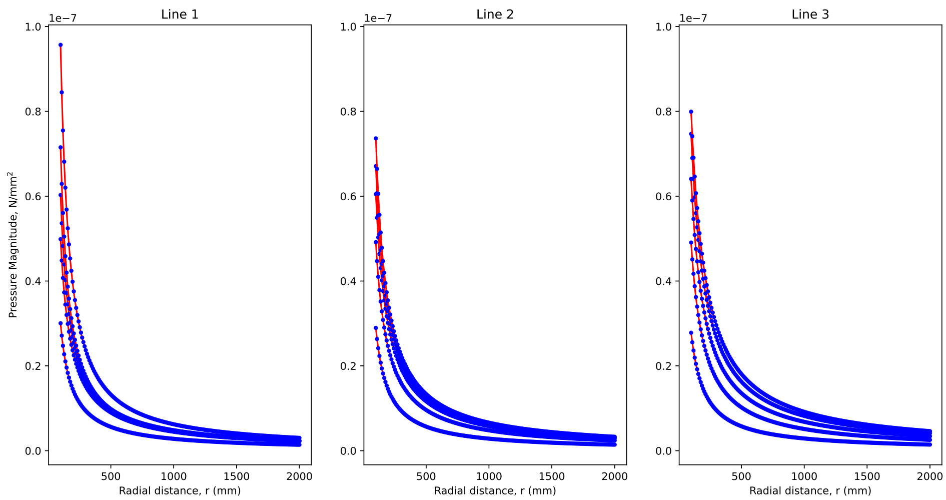

Figure 4: Sound pressure variation along the radial lines.

The z-component of acoustic particle velocity along `Line 1` is plotted in Figure 5, and shows excellent agreement, with a maximum difference of 0.28%.

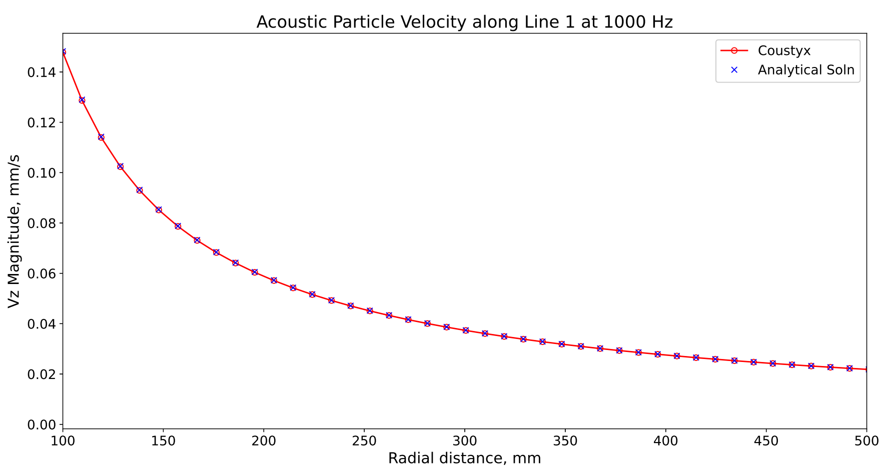

Figure 5: V_z variation along three radial lines.

6.1 Directivity

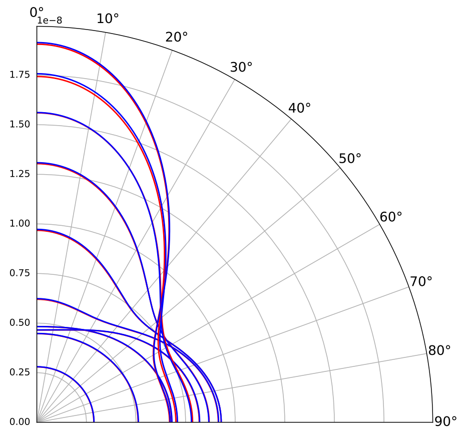

Figure 6: Sound pressure variation with polar angle θ at $r = 1000\,\mathrm{mm}$. Frequency varies from 500 Hz to 5000 Hz.

Figure 6 shows the directivity of the sound pressure from a hemispherical speaker in a rigid baffle. Analysis was performed from 500 Hz to 20 KHz in 500 Hz increments. Only the first ten frequencies are plotted in Figure 6. The blue line shows the analytical solution and the red line shows the Coustyx BEM solution. Excellent agreement is observed.

At lower frequencies, the pressure distribution is uniform (spherical wave). At intermediate frequencies, the pressure on the baffle plane is a little higher compared to the z-axis. At higher frequencies, the sound pressure levels are higher along the z-axis, as expected.

6.2 Sound Power

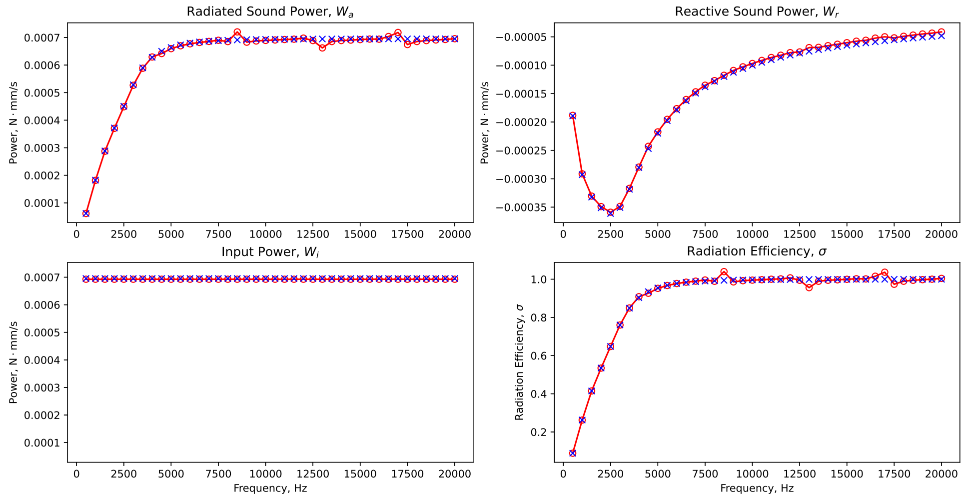

Figure 7: Variation of radiated power, reactive power, input power and radiation efficiency with frequency.

The acoustic variables related to sound power are plotted in Figure 7. There is a good agreement between the Coustyx BEM solution (in red) and the analytical solution (in blue). As expected, at higher frequencies, the speaker becomes more efficient with radiation efficiency σ approaching unity. The minor discrepancies seen in the radiated sound power at a few frequencies are related to the non-uniqueness or irregular frequency issue seen in Helmholtz BEM, remedied reasonably well by using five CHIEF points.