Oscillating Sphere Radiation Problem

1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model an oscillating sphere radiation problem using a MultiDomain model.
- Derive analytical solution for the exterior (unbounded) problem of the oscillating sphere.
- Validate Coustyx software by comparing Coustyx results to analytical solutions.

2 Model description

We model a sphere of radius $a = 1\,\mathrm{m}$. The fluid medium surrounding the sphere is air with sound speed $c = 343\,\mathrm{m/s}$ and mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\, Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c$. The sphere is oscillating with unit velocity in the z direction, that is $v_z = 1\,\mathrm{m/s}$. The BE mesh of the sphere is shown in Figure 1. Only an octant of the sphere is modeled by defining three planes of symmetry. One of the three planes is defined as an "Anti–symmetry plane". This is necessary to satisfy the boundary condition.

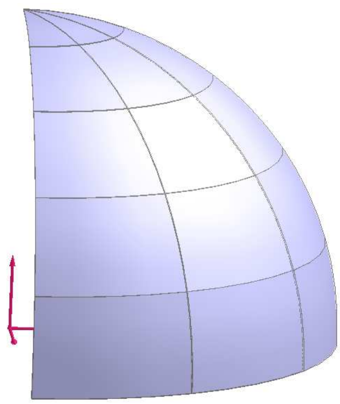

Figure 1: Boundary element mesh for an octant model of a sphere.

3 Boundary Conditions

The sphere is oscillating with unit velocity in the z direction. In Coustyx, this boundary condition is applied as an "Uniform Velocity", $v_z = 1$; where v_z is the component of velocity in the z direction.

4 Analytical solution

The oscillating sphere problem has exact solutions. For the exterior problem, the pressure at point (x,y,z) at a distance r from the center of the sphere is given by

$$\tilde{p}_{ext}(x, y, z) = 3v_z(ikZ_o)z\frac{h_1(kr)}{kr(h_0(ka) - h_2(ka))}$$

where h_l is the spherical Hankel function of order l. This expression can be reduced to

$$\tilde{p}_{ext}(x,y,z) = \frac{a^2}{r^3} v_z z (ikaZ_o) e^{ik(r-a)} (1-ikr) \frac{(k^2a^2 - 2 - 2ika)}{(k^4a^4+4)} \tag{1}$$

The velocity field corresponding to the exterior pressure field is

$$\vec{v}_{ext} = \tilde{v}_x \hat{e}_1 + \tilde{v}_y \hat{e}_2 + \tilde{v}_z \hat{e}_3$$

where the velocity field components are given by

$$\tilde{v}_x(x,y,z) = v_z \frac{zx}{r^3} \left[\frac{r(h_0(kr) - 2h_2(kr)) - 3h_1(kr)}{h_0(ka) - 2h_2(kr)} \right]$$

or,

$$\tilde{v}_x(x,y,z) = v_z (k^2r^2 - 3 + 3ikr) zx e^{ik(r-a)} a^3 \frac{(k^2a^2 - 2 - 2ika)}{r^5(k^4a^4+4)} \tag{2}$$

$$\tilde{v}_y(x,y,z) = v_z \frac{zy}{r^3} \left[\frac{r(h_0(kr) - 2h_2(kr)) - 3h_1(kr)}{h_0(ka) - 2h_2(kr)} \right]$$

or,

$$\tilde{v}_y(x,y,z) = v_z (k^2r^2 - 3 + 3ikr) zy e^{ik(r-a)} a^3 \frac{(k^2a^2 - 2 - 2ika)}{r^5(k^4a^4+4)} \tag{3}$$

$$\tilde{v}_z(x,y,z) = \frac{v_z}{h_0(ka) - 2h_2(kr)} \left[\frac{z^2}{r^2}(h_0(kr) - 2h_2(kr)) + \frac{3(x^2+y^2)}{r^3} h_1(kr) \right]$$

or,

$$\tilde{v}_z(x,y,z) = v_z (k^2r^2 - 3 + 3ikr) z^2 e^{ik(r-a)} a^3 \frac{(k^2a^2 - 2 - 2ika)}{r^5(k^4a^4+4)} + (1-ikr) e^{ik(r-a)} a^3 \frac{(k^2a^2 - 2 - 2ika)}{r^3(k^4a^4+4)} \tag{4}$$

The analytical expression for the radiated power by an oscillating sphere is given by

$$W = \frac{2\pi}{3} \rho_o c v_z^2 \frac{k^4a^6}{k^4a^4+4} \tag{5}$$

5 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 54.59Hz$ using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".

Coustyx uses Direct BE method to solve the radiation problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Field point solutions are then computed from the surface solutions.

The surface pressure on the sphere at point (0, 0, 1) from Coustyx is (82.68, -249.33), which compares well with the analytical solution (83.01, -249.02). Figure 2 shows field point pressure variation with distance from both Coustyx and Analytical methods. The comparisons show very good agreement between the two solutions. The radiated power computed from Coustyx is 171.94 Watts, which matches well with the exact analytical solution of 173.85 Watts.

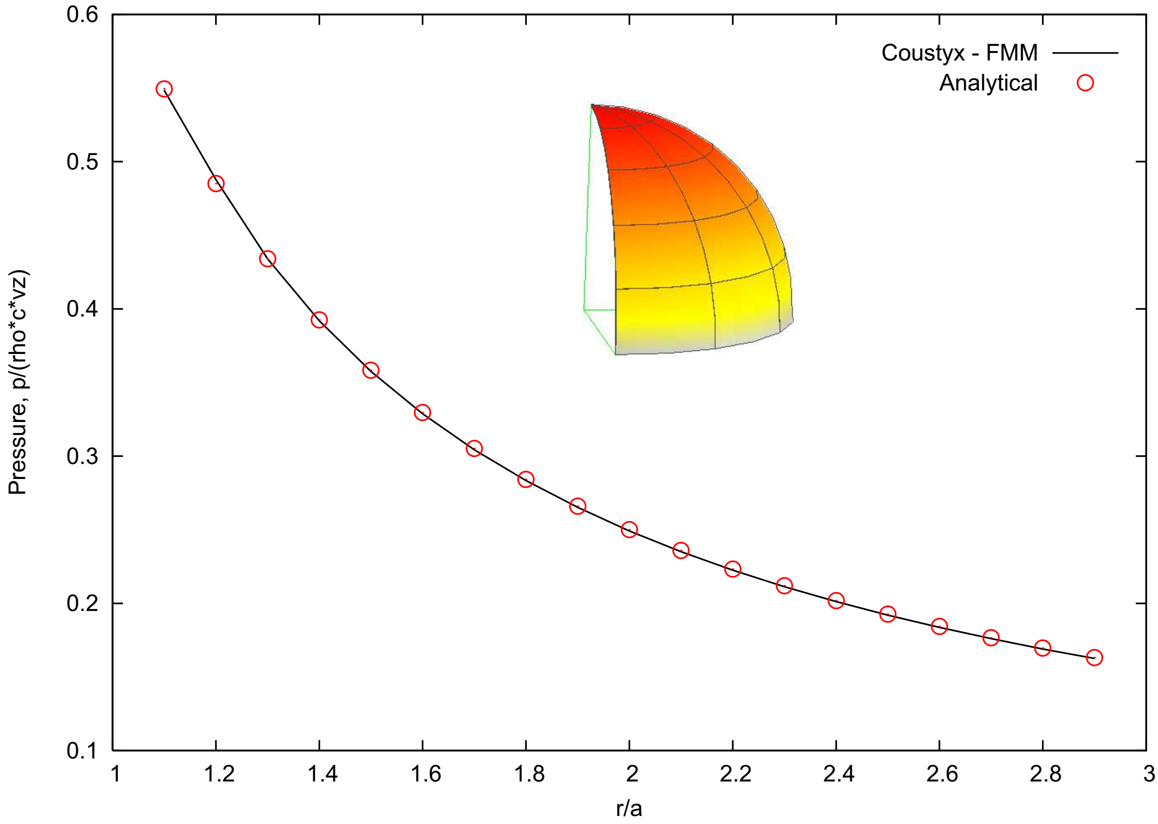

Figure 2: Field point pressure comparisons with distance, r (from center of sphere), from Coustyx and Analytical methods.