Exterior Piston Baffled in a Sphere

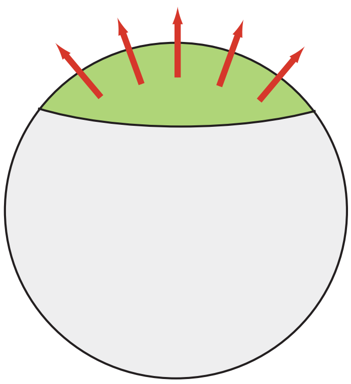
1 Introduction

The main objectives of this Demo Model are

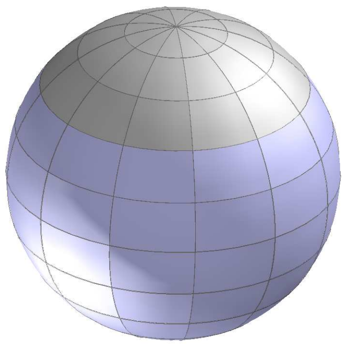
- Demonstrate the ability of Coustyx to model an exterior piston baffled in a sphere using a MultiDomain model.
- Derive analytical solution for the exterior (unbounded) problem of the spherical baffle radiation problem.
- Validate Coustyx software by comparing Coustyx results to analytical solutions.

2 Model description

We model a sphere of radius $a = 1\,\text{m}$. The fluid medium surrounding the sphere is air with sound speed $c = 343\,\text{m/s}$ and mean density $\rho_o = 1.21\,\text{kg/m}^3$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\,Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c$. A uniform radial velocity $v_r = 1\,\text{m/s}$ is applied on the sphere from $\theta = 0$ to $\theta = 54^o$ to simulate an exterior piston baffled in the sphere. The BE mesh of the spherical baffle is shown in Figure 1.



(a) Spherical baffle



(b) Boundary element mesh

Figure 1: Acoustic problem description.

3 Boundary Conditions

The portion of the sphere between $\theta = 0 - 54^o$ is vibrating with a uniform outward radial velocity $v_r = 1$, and the rest of the sphere is assumed to be rigid. In Coustyx, this boundary condition is applied as an “Uniform Normal Velocity”, $v_n = -v_r$; where v_n is the velocity in the direction of *Domain Normal*. Note that all boundary conditions in a MultiDomain model are defined with respect to the *Domain Normal*, which always points away from the domain of interest. For the spherical baffle radiation problem, the exterior domain is our domain of interest; hence, domain normal is pointing away from the exterior domain, that is, it is pointing into the sphere.

4 Analytical solution

The exterior spherical baffle with a uniform normal velocity has analytical solutions. The analytical solution to the Helmholtz equation can be expressed as a series of spherical harmonics

$$p(r,\theta,\phi) = \sum_{l=0}^{N} A_l P_l(\cos\theta)\cos(m\phi) h_l^1(kr) \tag{1}$$

where P_l is the Legendre polynomial of degree l, $h_l^1(kr)$ is the spherical Hankel function of the first kind of order l.
The velocity boundary condition is

$$v_n(\theta,\phi) = \begin{cases} u_0 & 0 \le \theta < \theta_0 \\ 0 & \theta_0 < \theta \le \pi \end{cases} \tag{2}$$

The velocity distribution can be represented as a series of spherical harmonics

$$\begin{aligned} v_n(\theta) = -v_r(\theta) &= \sum_{l=0}^{N} u_l P_l(\cos\theta)\cos(m\phi) \\ u_l &= (l+\tfrac{1}{2}) \int_0^{\pi} v_n(\theta) P_l(\cos\theta) \sin\theta d\theta \end{aligned} \tag{3}$$

The recurrence formulas of Legendre polynomials are used to solve for u_l,

$$u_l = (l+\frac{1}{2}) u_0 \int_0^{\theta_0} P_l(\cos\theta)\sin\theta d\theta = \frac{1}{2} u_0 \left[P_{l-1}(\cos\theta_0) - P_{l+1}(\cos\theta_0)\right] \tag{4}$$

where for case $l = 0$, we consider $P_{-1}(x) = 1$.
The above expression for u_l is substituted in Equation 3 to obtain series expansion for velocity on the surface of the sphere. The coefficient A_l in Equation 1 is determined by matching the radial velocity from the assumed solution with the specified normal velocity. Thus from Equation 1

$$\frac{\partial p}{\partial r}(r,\theta,\phi) = \sum_{l=0}^{N} A_l P_l^m(\cos\theta)\cos(m\phi) \left[\frac{kl(h_{l-1}^1(kr) - h_{l+1}^1(kr)) - kh_{l+1}^1(kr)}{(2l+1)}\right] \tag{5}$$

The radial velocity on the sphere (at $r = a$) is related to the pressure gradient in the radial direction as

$$v_r(\theta,\phi)(ikZ_0) = \frac{\partial p}{\partial r}(a,\theta,\phi) \tag{6}$$

Using the orthogonal properties of Legendre polynomials we obtain,

$$A_l = \frac{-(ikZ_0)(2l+1)u_l}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)} \tag{7}$$

Therefore, the pressure at a field point (r,θ,ϕ) in the exterior domain is derived to be

$$p(r,\theta,\phi) = \sum_{l=0}^{N} \left[\frac{-(ikZ_0)(2l+1)u_l}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right] P_l(\cos\theta)\cos(m\phi) h_l^1(kr) \tag{8}$$

The velocity at the point(r,θ,ϕ) is

$$\overrightarrow{v}(r,\theta,\phi) = 1/(ikZ_0)\overrightarrow{\nabla} p(r,\theta,\phi) \tag{9}$$

$$v_r(r,\theta,\phi) = -\sum_{l=0}^{N} \left[\frac{klh_{l-1}(kr) - klh_{l+1}(kr) - kh_{l+1}(kr)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right] u_l P_l(\cos\theta)\cos(m\phi) \tag{10}$$

$$v_\theta(r,\theta,\phi) = (1/r)\sum_{l=0}^{N} \left[\frac{h_l(kr)u_l(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right] \left[\frac{lP_{l-1}(\cos\theta) - l\cos\theta P_l(\cos\theta)}{\sin\theta}\right] \cos(m\phi) \tag{11}$$

$$v_\phi(r,\theta,\phi) = (m/r)\sum_{l=0}^{N}\left[\frac{h_l(kr)u_l(2l+1)}{klh_{l-1}(ka)-klh_{l+1}(ka)-kh_{l+1}(ka)}\right]\frac{P_l(\cos\theta)}{\sin\theta}\sin(m\phi)$$
$$v_\phi = 0, \quad \text{for } m=0 \tag{12}$$

The pressure and velocities in Cartesian coordinates can be obtained from the following transformations:

$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \arccos(z/\sqrt{x^2+y^2+z^2}), 0 \le \theta \le \pi \tag{13}$$
$$\phi = \arctan(y/x), 0 \le \phi \le 2\pi$$

$$p(x,y,z) = p(r(x,y,z), \theta(x,y,z), \phi(x,y,z)) \tag{14}$$

$$v_x(x,y,z) = \frac{x}{r}v_r + \frac{xz}{r\sqrt{x^2+y^2}}v_\theta - \frac{y}{\sqrt{x^2+y^2}}v_\phi \tag{15}$$

$$v_y(x,y,z) = \frac{y}{r}v_r + \frac{yz}{r\sqrt{x^2+y^2}}v_\theta + \frac{x}{\sqrt{x^2+y^2}}v_\phi \tag{16}$$

$$v_z(x,y,z) = \frac{z}{r}v_r - \frac{\sqrt{x^2+y^2}}{r}v_\theta \tag{17}$$

Acoustic intensity, I, is the time average of the rate of sound energy flow per unit area normal to the direction of propagation of the wave. It is a vector quantity in the direction of velocity. For time-harmonic waves, where the time dependence of pressure and velocity can be represented by $e^{-i\omega t}$, the intensity reduces to

$$I = \frac{1}{T}\int_0^T PV dt = \frac{1}{2}\mathrm{Re}\{pv^*\} \tag{18}$$

where * denotes the complex conjugate and Re indicates the real part.
Using the orthogonality properties of the Legendre polynomials, the analytical expression for radiated power W by the spherical baffle is derived to be

$$W = \int_S I_n dS = 2\pi a^2 \mathrm{Re}\left\{\sum_{l=0}^{N}\frac{iZ_0 h_l(ka)u_l^2}{lh_{l-1}(ka) - lh_{l+1}(ka) - h_{l+1}(ka)}\right\} \tag{19}$$

The radiation efficiency σ is defined as

$$\sigma = \tfrac{W}{\Pi}$$
$$\Pi = Z_0\tfrac{1}{2}\int_S v_n^2 dS = \pi a^2 Z_0 u_0^2(1-\cos\theta_0) \tag{20}$$

where Π is the input power.
Therefore, the analytical expression for radiation efficiency is

$$\sigma = \frac{1}{2(1-\cos\theta_0)}\mathrm{Re}\left\{\sum_{l=0}^{N}\frac{ih_l(ka)(P_{l-1}(\cos\theta_0) - P_{l+1}(\cos\theta_0))^2}{lh_{l-1}(ka) - lh_{l+1}(ka) - h_{l+1}(ka)}\right\} \tag{21}$$

5 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 54.59Hz$ using the Fast Multipole Method (FMM) by running “Run Validation - FMM”. Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".

Coustyx uses Direct BE method to solve the radiation problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Field point solutions are then computed from the surface solutions.

Figure 2 shows comparisons of field point pressures computed from both Coustyx and analytical methods. The specified field points are located at (r_f, θ_f, ϕ_f), where $r_f = 1.5\,\mathrm{m}, \phi_f = 0$ and $\theta_f = i\pi/20$, $i = 0, ..., 20$. The comparisons show very good agreement between the solutions computed from Coustyx and analytical expressions. The radiated power computed by Coustyx, 97.83 Watts, matches well with the analytical solution, 98.64 Watts.

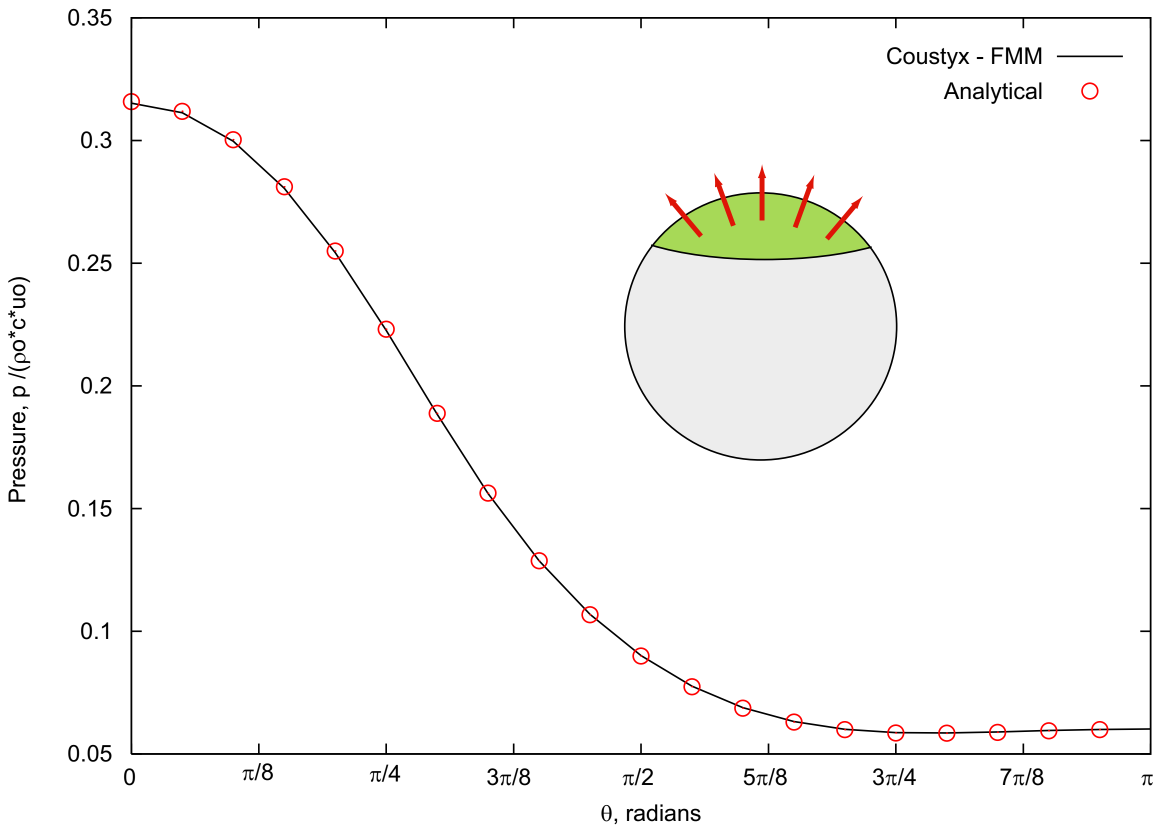


Figure 2: Field point pressure comparisons for a spherical baffle from Coustyx and analytical methods.