Radiation from a Sphere with Spherical Harmonic Excitation

1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model the radiation problem of a sphere with spherical harmonic excitation using a MultiDomain model.
- Demonstrate the ability of Coustyx to define complex boundary conditions using *scripts*.
- Define CHIEF points in the sphere interior to suppress large errors due to non-uniqueness issue at eigen-frequencies of the corresponding interior problem.
- Validate Coustyx program by comparing the results from Coustyx to the analytical solutions.

2 Model description

We model a sphere of radius $a = 1\,\mathrm{m}$. The fluid medium around the sphere is air with sound speed $c = 343\,\mathrm{m/s}$ and mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\,Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c$. The BE mesh of the sphere is shown in Figure 1.

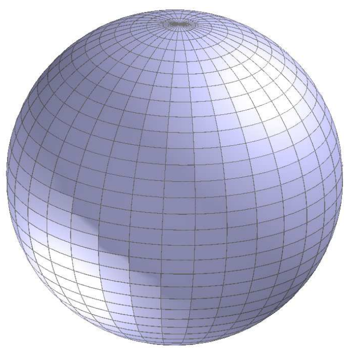

Figure 1: Boundary element mesh of a sphere with unit radius.

A radial velocity distribution, $\tilde{v}_r$, represented by arbitrary spherical harmonics is applied on the sphere,

$$\tilde{v}_r(\theta, \phi) = -u_0 P_l^m(\cos\theta)\cos(m\phi) \tag{1}$$

where P_l^m is the associated Legendre function of degree $l = 4$ and order $m = 2$, and $u_0 = 1$ is a scalar coefficient.

3 Boundary Conditions

The sphere is vibrating with a radial velocity $\tilde{v}_r$ given by spherical harmonics (see Equation 1). In Coustyx, this boundary condition is applied as an “Nonuniform Normal Velocity” and is defined by script. The normal velocity, $v_n = -\tilde{v}_r$, is the velocity in the direction of *Domain Normal.* Note that all boundary conditions in a MultiDomain model are defined with respect to the *Domain Normal*, which always points away from the domain of interest. For this example, the exterior domain is our domain of interest; hence, domain normal is pointing away from the exterior domain, that is, it is pointing into the sphere.

4 CHIEF points

In Direct BEM radiation problems, CHIEF (Combined Helmholtz Integration Equation Formulation) points are used to eliminate large errors in the solution at certain frequencies. These frequencies are the eigen-frequencies of the corresponding interior problem. At these frequencies the Helmholtz equation doesn't yield a unique solution. We define a few arbitrary CHIEF points inside the sphere to additionally constraint the problem and obtain accurate solutions. Please note that when a CHIEF point falls on an interior nodal surface it provides no additional constraint effect. Hence, selection of good CHIEF points is crucial in obtaining accurate solutions at all frequencies. One way to ensure that is to define more than one CHIEF points at random locations inside the sphere.

5 Analytical solution

The exact solution to the Helmholtz equation in the exterior domain can be assumed to be of the form

$$p(r,\theta,\phi) = A_l^m u_0 P_l^m(\cos\theta)\cos(m\phi)h_l^1(kr) \tag{2}$$

where $h_l^1(kr)$ is the spherical Hankel function of the first kind of order l and A_l^m is a constant dependent on (l,m). We need to solve for A_l^m to get analytical expression for pressure in the exterior domain.
The pressure gradient in the radial direction on the surface of the sphere is

$$\frac{\partial p}{\partial r}(r,\theta,\phi) = A_l^m u_0 P_l^m(\cos\theta)\cos(m\phi)\left[\frac{kl(h_{l-1}^1(kr) - h_{l+1}^1(kr)) - kh_{l+1}^1(kr)}{(2l+1)}\right] \tag{3}$$

The specified radial velocity ($\tilde{v}_r$) and the pressure gradient in the radial direction on the surface of the sphere ($r=a$) are related and can be used to obtain A_l^m, that is,

$$\tilde{v}_r(\theta,\phi)(ikZ_0) = \frac{\partial p}{\partial r}(a,\theta,\phi) \tag{4}$$

$$A_l^m = \frac{-(ikZ_0)(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)} \tag{5}$$

Thus, the analytical expression for pressure at any point (r,θ,ϕ) in the exterior domain is given by

$$p(r,\theta,\phi) = \left[\frac{-(ikZ_0)(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right]u_0 P_l^m(\cos\theta)\cos(m\phi)h_l^1(kr) \tag{6}$$

The velocity at the exterior point (r,θ,ϕ) is

$$\overrightarrow{v}(r,\theta,\phi) = 1/(ikZ_0)\overrightarrow{\nabla}p(r,\theta,\phi) \tag{7}$$

$$v_r(r,\theta,\phi) = -u_0\left[\frac{klh_{l-1}(kr) - klh_{l+1}(kr) - kh_{l+1}(kr)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right]P_l^m(\cos\theta)\cos(m\phi) \tag{8}$$

$$v_\theta(r,\theta,\phi) = (1/r)u_0\left[\frac{h_l(kr)(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right]\left[\frac{lP_{l-1}(\cos\theta) - l\cos\theta P_l^m(\cos\theta)}{\sin\theta}\right]\cos(m\phi) \tag{9}$$

$$v_\phi(r,\theta,\phi) = (m/r)u_0\left[\frac{h_l(kr)(2l+1)}{klh_{l-1}(ka) - klh_{l+1}(ka) - kh_{l+1}(ka)}\right]\frac{P_l^m(\cos\theta)}{\sin\theta}\sin(m\phi) \tag{10}$$

where v_r, v_θ and v_ϕ are the components of velocity in the spherical coordinate system.
The pressure and velocity in Cartesian coordinates can be obtained by applying the following transformations:

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arccos(z/\sqrt{x^2+y^2+z^2}), 0 \le \theta \le \pi \\ \phi &= \arctan(y/x), 0 \le \phi \le 2\pi \end{aligned} \tag{11}$$

$$p(x,y,z) = p(r(x,y,z), \theta(x,y,z), \phi(x,y,z)) \tag{12}$$

$$v_x(x,y,z) = \frac{x}{r}v_r + \frac{xz}{r\sqrt{x^2+y^2}}v_\theta - \frac{y}{\sqrt{x^2+y^2}}v_\phi \tag{13}$$

$$v_y(x,y,z) = \frac{y}{r}v_r + \frac{yz}{r\sqrt{x^2+y^2}}v_\theta + \frac{x}{\sqrt{x^2+y^2}}v_\phi \tag{14}$$

$$v_z(x,y,z) = \frac{z}{r}v_r - \frac{\sqrt{x^2+y^2}}{r}v_\theta \tag{15}$$

The pressure on the exterior surface of the sphere can be written in terms of the spherical harmonic excitation on the exterior surface as

$$p(a,\theta,\varphi) = f.(ikz_o)v_r(\theta,\varphi) = (f_r + if_i).(ikz_o)v_r(\theta,\varphi) = (kz_o)(-f_i + f_r)v_r(\theta,\varphi)$$

where the factor $f = (f_r + if_i)$ is

$$f = \frac{(2l+1)h_l^1(ka)}{\left[kl(h_{l-1}^1(ka) - h_{l+1}^1(ka)) - kh_{l+1}^1(ka)\right]}$$

Using the orthogonality of the associated Legendre functions, that is,

$$\int_{-1}^{1} [P_l^m(x)]^2 dx = 1$$

the integral over the norm of the radial velocity is reduced to

$$\begin{aligned}\int_S |v_r|^2 dS &= \int\int u_0^2 [P_l^m(\cos\theta)]^2 [\cos^2 m\phi] a^2 \sin\theta d\theta d\varphi \\ &= C_m \pi a^2 u_0^2\end{aligned}$$

where $C_m = 1$ for $m \neq 0$, and $C_m = 2$ for $m = 0$.
The analytical expression for the radiated power (W) due to spherical harmonic excitation on a sphere is derived to be,

$$\begin{aligned}W &= \tfrac{1}{2}\mathrm{Re}\{\int_S pv^* dS\} \\ &= -\tfrac{1}{2}kz_o f_i \int_S |v_r|^2 dS \\ &= -C_m \tfrac{1}{2}kz_o f_i \pi a^2 u_0^2\end{aligned} \tag{16}$$

where $C_m = 1$ for $m \neq 0$, and $C_m = 2$ for $m = 0$.
The radiation efficiency σ_0 is given by,

$$\sigma_0 = \frac{W}{\frac{1}{2}\int_S z_o v_r^2(\theta,\varphi) dS} = -kf_i \tag{17}$$

6 Results and validation

Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In the demo model, the analysis is performed for the frequency range $ka = 0.2$ to $ka = 10$ with a resolution $\Delta = 0.2$ using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".
Coustyx uses Direct BE method to solve the radiation problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Radiated sound power is then computed at all frequencies.
Figure 2 shows comparisons of radiated sound power computed from both Coustyx and analytical methods for $ka = 0.2$ to $ka = 10$. The comparisons show very good agreement between the two methods. From the Figure 2 one can see that errors due to non-uniqueness problem at $ka = 5.6$

and $ka = 9.8$ are suppressed in Coustyx solution due to the use of CHIEF points. The existence of the non-uniqueness problem at these frequencies can be verified by deleting all CHIEF points and running the analysis once again.

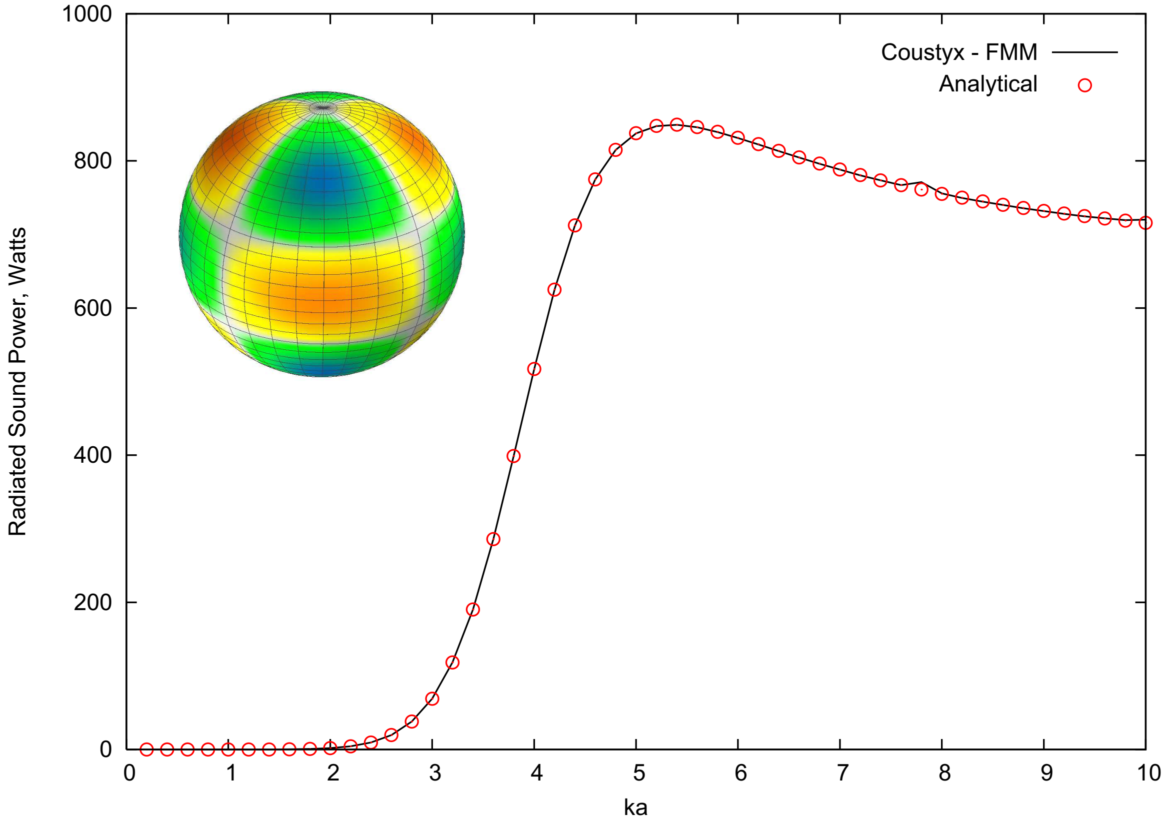

Figure 2: Radiated sound power comparisons for a sphere with spherical harmonic excitation.