Plane Wave Scattering by a Rigid Sphere

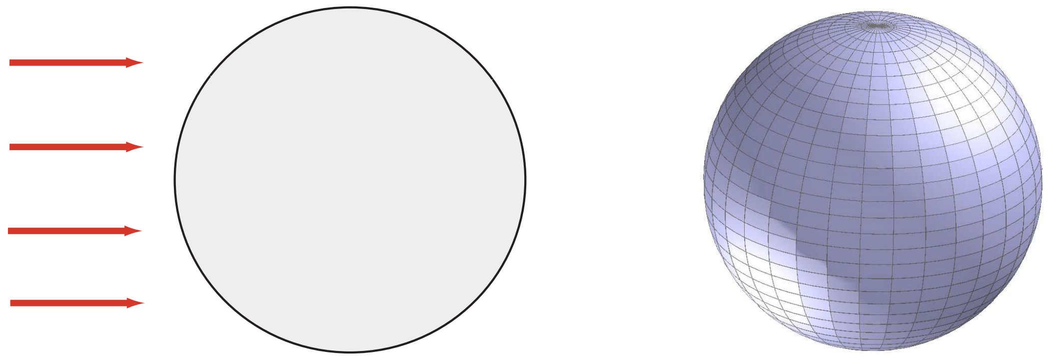
1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model the scattering of a plane wave by a rigid sphere using MultiDomain model.
- Derive analytical solution.
- Validate Coustyx software by comparing Coustyx results to the analytical solutions.

2 Model description

We model a sphere of radius $a = 1\,\mathrm{m}$. The fluid medium surrounding the sphere is air with sound speed $c = 343\,\mathrm{m/s}$ and mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\,Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c$. A plane wave of amplitude $p_o = 1$ propagating in the direction of $+z$ direction is scattered by the rigid sphere centered at the origin $(0, 0, 0)$. The sphere in the presence of the plane wave is shown in Figure 1.



(a) Plane wave scattered by a rigid sphere

(b) Boundary element mesh

Figure 1: Acoustic problem description.

3 Boundary Conditions

In Coustyx MultiDomain model the rigid boundary condition on the sphere is applied using "Uniform Normal Velocity" type with zero amplitude. Note that all boundary conditions in a MultiDomain model are defined with respect to the *Domain Normal*, which always points away from the domain of interest. For this problem, the exterior domain is our domain of interest; hence, domain normal is pointing away from the exterior domain, that is, it is pointing into the sphere.
The plane wave source is introduced in the Coustyx model by defining a new acoustic source of "Plane Wave" type of amplitude $p_o = 1$, with origin at $(0, 0, 0)$ and direction $(0, 0, 1)$.

4 Analytical solution

An incident plane wave of amplitude p_o traveling in $+z$ direction is given by

$$p_{inc} = p_o e^{ikz} \tag{1}$$

The plane wave is impinging on the rigid sphere centered at the origin. This plane wave can be represented as a series of spherical harmonics as follows (refer [1]),

$$p_{inc} = p_o \sum_{l=0}^{N} (2l+1) i^l j_l(kr) P_l(\cos\theta) \tag{2}$$

where j_l is the spherical Bessel function, P_l is the Legendre function of order l, and $i = \sqrt{-1}$.
Assume $p_{scattered}$ is the scattered sound pressure by the rigid sphere. The function $p_{scattered}$ should be selected such that it satisfies both the Helmholtz wave equation and the Sommerfeld radiation condition. Hence, the scattered wave can be represented as follows,

$$p_{scattered} = \sum_{l=0}^{N} A_l h_l(kr) P_l(\cos\theta) \tag{3}$$

where h_l is the spherical Hankel function of the first kind of order l, A_l is the coefficient of interest. The coefficient A_l is determined from setting the normal derivative of the scattered pressure on the sphere surface to be opposite of the normal derivative of the incident pressure on the surface of the sphere, so that the total normal velocity on the sphere surface is zero (rigid boundary condition). That is,

$$\frac{\partial p_{inc}}{\partial r} + \frac{\partial p_{scattered}}{\partial r} = 0, \text{ at r = a} \tag{4}$$

Therefore,

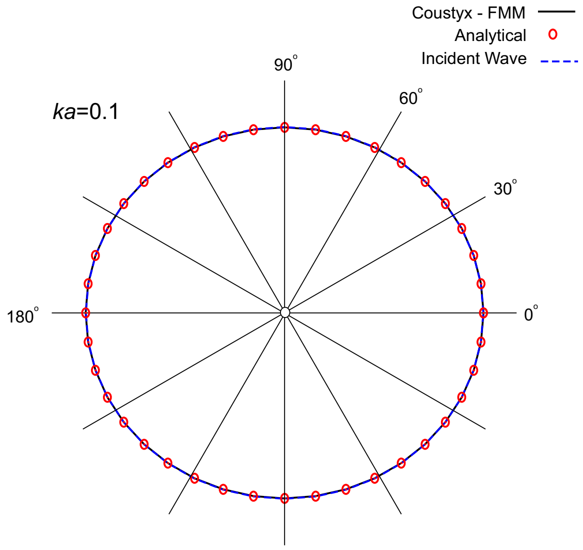
$$A_l = -p_o(2l+1) i^l \frac{l j_{l-1}(ka) - (l+1) j_{l+1}(ka)}{l h_{l-1}(ka) - (l+1) h_{l+1}(ka)} \tag{5}$$

The total sound pressure at any field point is the sum of the incident and scattered pressures,

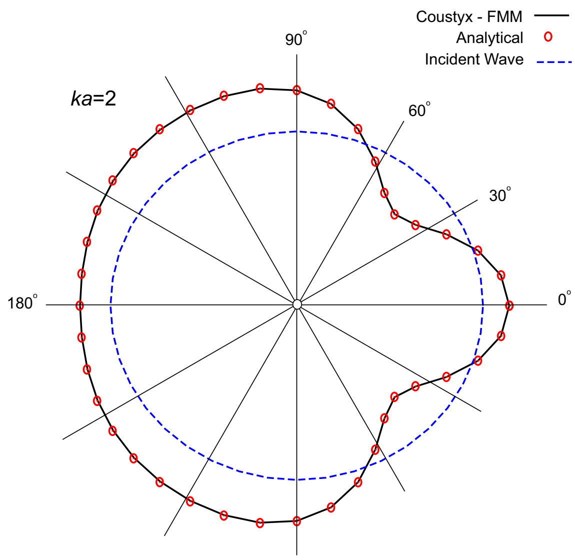
$$p_{total} = p_{inc} + p_{scattered} \tag{6}$$

5 Results and validation

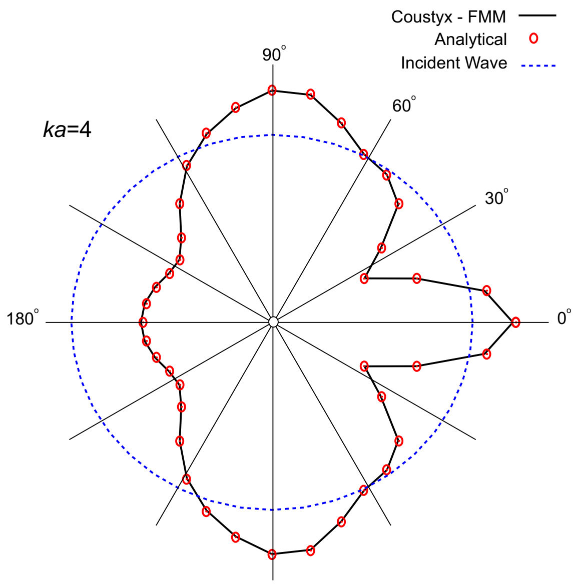
Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 109.18Hz$ (that is, $ka = 2$) using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".
Coustyx uses Direct BE method to solve the acoustic problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Field point solutions are then computed from the surface solutions.
Figure 2 shows angular distribution of the field point pressure amplitudes at $r = 1.5$ computed from both Coustyx and analytical methods for different values of ka. The quantities are plotted versus the polar angle θ. $\theta = 180^o$ corresponds to the front end and $\theta = 0^o$ corresponds to the back end of the sphere with respect to the impinging plane wave. The comparisons between the solutions computed from Coustyx and analytical expressions show very good agreement. Note that when the wavelength of the plane wave is very large compared to the radius of the sphere a, that is $ka << 1$, the pressure field does not vary much by the presence of the sphere. Less or no scattering occurs for this case as seen in Figure 2 for $ka = 0.1$, where the total pressure field from Coustyx or analytical solution overlaps with the incident wave pressure field (implying that the scattered field is nearly zero). More scattering occurs with the increase in ka. This is illustrated in Figure 2 for $ka = 2$ and $ka = 4$.



(a) ka=0.1



(b) ka=2



(c) ka=4

Figure 2: Angular distribution of total field point pressure amplitudes at $r = 1.5a$ due to a plane wave scattered by a rigid sphere of radius a from Coustyx and analytical methods. The incident wave pressure amplitude p_o is also plotted for reference. The quantities are plotted versus the polar angle θ for different values of ka.

References

[1] J. W. S Rayleigh. *The Theory of Sound - Volume II.* Dover, 1945. Page 272.