Spherical Wave Scattering by a Rigid Sphere

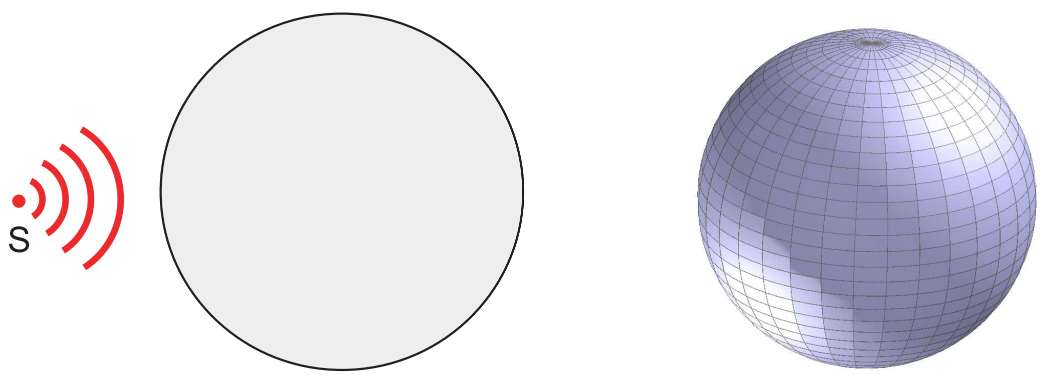
1 Introduction

The main objectives of this Demo Model are

- Demonstrate the ability of Coustyx to model the scattering of a spherical wave by a rigid sphere using MultiDomain model.
- Derive analytical solution.
- Validate Coustyx software by comparing Coustyx results to the analytical solutions.

2 Model description

We model a sphere of radius $a = 1\,\mathrm{m}$. The fluid medium surrounding the sphere is air with sound speed $c = 343\,\mathrm{m/s}$ and mean density $\rho_o = 1.21\,\mathrm{kg/m^3}$. The characteristic impedance of air $Z_o = \rho_o c = 415.03\,Rayl$. The wavenumber at a frequency ω is given as $k = \omega/c$. A spherical wave source (or a monopole source) of unit volume velocity, $q = 1$, is located at $(0, 0, -3)$. The spherical wave is scattered by the rigid sphere centered at the origin $(0, 0, 0)$. The acoustic problem is illustrated in Figure 1.



(a) Spherical wave scattered by a rigid sphere (b) Boundary element mesh

Figure 1: Acoustic problem description.

3 Boundary Conditions

In Coustyx MultiDomain model the rigid boundary condition on the sphere is applied using "Uniform Normal Velocity" type with zero amplitude. Note that all boundary conditions in a MultiDomain model are defined with respect to the *Domain Normal*, which always points away from the domain of interest. For this problem, the exterior domain is our domain of interest; hence, domain normal is pointing away from the exterior domain, that is, it is pointing into the sphere.
The spherical wave source is introduced in the Coustyx model by defining a new acoustic source of "Monopole" type with unit amplitude located at $(0, 0, -3)$.

4 Analytical solution

The acoustic field pressure p_{inc} at any point $\mathbf{r_o}(r_o, \theta_o, \phi_o)$ due to a spherical wave source of strength $\varpi = ikZ_oq$ located at $\mathbf{r}_s(r_s, \theta_s, \phi_s)$ is

$$p_{inc}(\mathbf{r_o}) = -\varpi G_\infty(\mathbf{r_o} - \mathbf{r_s}) \tag{1}$$

where $G_\infty = \frac{e^{ikr}}{4\pi r}$ is the free-space Green's function and $\vec{r} = \left\{ \begin{array}{c} x_o - x_s \\ y_o - y_s \\ z_o - z_s \end{array} \right\}; r = \|\vec{r}\|$.
The spherical wave is impinging on the rigid sphere centered at the origin.
In order to solve the scattering problem we need to expand the free-space Green's function in spherical harmonics.

$$G_\infty = \frac{ik}{2\pi} \sum_{l=0}^{\infty} \sum_{m=0}^{+l} \varepsilon_m \cos m(\phi - \phi_s) p_l^m(\cos\theta_s) p_l^m(\cos\theta) j_l(kr_<) h_l(kr_>) \tag{2}$$

where j_l is the spherical Bessel function, h_l is the spherical Hankel function of the first kind, p_l^m is the *normalized* Associated Legendre function of the first kind, $i = \sqrt{-1}$, and,

$$\begin{array}{l} r_< = \min(r, r_s) \\ r_> = \max(r, r_s) \\ \varepsilon_m = \left\{ \begin{array}{l} 1, m = 0 \\ 2, m \neq 0 \end{array} \right. \end{array}$$

Assume $p_{scattered}$ is the scattered sound pressure by the rigid sphere. The function $p_{scattered}$ should be selected such that it satisfies both the Helmholtz wave equation and the Sommerfeld radiation condition. Hence, the scattered wave at a point $\mathbf{r}(r, \theta, \phi)$ is represented as follows,

$$p_{scattered} = \sum_{l=0}^{\infty} \sum_{m=0}^{+l} h_l(kr) p_l^m(\cos\theta) \left[A_{l,m} \cos m\phi + B_{l,m} \sin m\phi\right] \tag{3}$$

where $A_{l,m}$ and $B_{l,m}$ are the coefficients of interest.
The coefficients $A_{l,m}$ and $B_{l,m}$ are determined from setting the normal derivative of the scattered pressure on the sphere surface to be opposite of the normal derivative of the incident pressure on the surface of the sphere, so that the total normal velocity on the sphere surface is zero (rigid boundary condition). That is,

$$\frac{\partial p_{inc}}{\partial r} + \frac{\partial p_{scattered}}{\partial r} = 0, \text{ at r = a} \tag{4}$$

The spherical wave source is located outside the sphere, that is $r_s > a$. Hence for any point $\mathbf{r}(r, \theta, \phi)$ on the sphere, $r_< = r(= a)$, $r_> = r_s$. Therefore,

$$\begin{array}{rl} \left.\frac{\partial p_{inc}}{\partial r}\right|_{r=a} & = -\varpi \frac{ik}{2\pi} \sum_{l=0}^{\infty} \sum_{m=0}^{+l} \varepsilon_m \cos m(\phi - \phi_s) p_l^m(\cos\theta_s) p_l^m(\cos\theta) h_l(kr_s) j_l'(ka) \\ \left.\frac{\partial p_{scattered}}{\partial r}\right|_{r=a} & = \sum_{l=0}^{\infty} \sum_{m=0}^{+l} p_l^m(\cos\theta) \left[A_{l,m} \cos m\phi + B_{l,m} \sin m\phi\right] h_l'(ka) \end{array} \tag{5}$$

where $(2l+1)j_l'(kr) = klj_{l-1}(kr) - k(l+1)j_{l+1}(kr)$ and $(2l+1)h_l'(kr) = klh_{l-1}(kr) - k(l+1)h_{l+1}(kr)$ and $' \equiv \frac{\partial}{\partial r}$.
Substituting Equation 5 into Equation 4 we obtain the coefficients $A_{l,m}$ and $B_{l,m}$.

$$\begin{array}{l} A_{l,m} = \varepsilon_m \varpi \frac{ik}{2\pi} \left[\frac{lj_{l-1}(ka) - (l+1)j_{l+1}(ka)}{lh_{l-1}(ka) - (l+1)h_{l+1}(ka)}\right] \cos m\phi_s p_l^m(\cos\theta_s) h_l(kr_s) \\ B_{l,m} = \varepsilon_m \varpi \frac{ik}{2\pi} \left[\frac{lj_{l-1}(ka) - (l+1)j_{l+1}(ka)}{lh_{l-1}(ka) - (l+1)h_{l+1}(ka)}\right] \sin m\phi_s p_l^m(\cos\theta_s) h_l(kr_s) \end{array}$$

Therefore, the scattered sound pressure at any field point is

$$p_{scattered} = \varpi \frac{ik}{2\pi} \sum_{l=0}^{\infty} \sum_{m=0}^{+l} \varepsilon_m \left[\frac{lj_{l-1}(ka) - (l+1)j_{l+1}(ka)}{lh_{l-1}(ka) - (l+1)h_{l+1}(ka)}\right] \cos m(\phi - \phi_s) p_l^m(\cos\theta_s) p_l^m(\cos\theta) h_l(kr_s) h_l(kr) \tag{6}$$

The total sound pressure at any field point is the sum of the incident and scattered pressures,

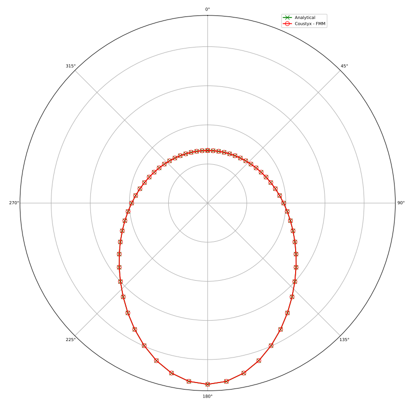
$$p_{total} = p_{inc} + p_{scattered} \tag{7}$$

5 Results and validation

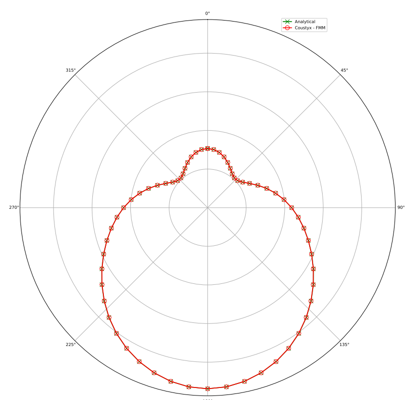
Acoustic analysis is carried out by running one of the Analysis Sequences defined in the Coustyx MultiDomain model. An Analysis Sequence stores all the parameters required to carry out an analysis, such as frequency of analysis, solution method to be used, etc. In this Demo Model, the analysis is performed at a frequency $f = 109.18Hz$ (that is, $ka = 2$) using the Fast Multipole Method (FMM) by running "Run Validation - FMM". Coustyx analysis results, along with the analytical solutions, are written to the output file "validation_results_fmm.txt". The results can be plotted using the matlab file "PlotResults.m".

Coustyx uses Direct BE method to solve the acoustic problem. In Direct BE method, the primary variables are the pressure and the pressure gradient on the boundary. Field point solutions are then computed from the surface solutions.

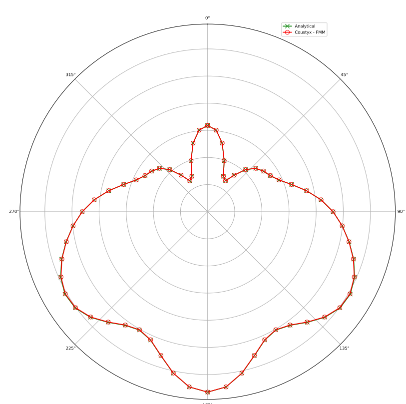
Figure 2 shows angular distribution of the field point pressure amplitudes at $r = 1.5$ computed from both Coustyx and analytical methods for different values of ka. The quantities are plotted versus the polar angle θ. $\theta = 180^o$ corresponds to the front end and $\theta = 0^o$ corresponds to the back end of the sphere with respect to the impinging spherical wave. The comparisons between the solutions computed from Coustyx and analytical expressions show very good agreement.



(a) ka=0.1



(b) ka=2



(c) ka=4

Figure 2: Angular distribution of total field point pressure amplitudes at $r = 1.5a$ due to a spherical wave scattered by a rigid sphere of radius a from Coustyx and analytical methods. The quantities are plotted versus the polar angle θ for different values of ka.