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LINEARIZATION OF MULTIBODY FRICTIONAL CONTACT PROBLEMS

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Abstract—A Simplex type algorithm is used to impose frictional contact conditions on finite element models of bodies that move close to trajectories that can be determined from kinematic constraints on the bodies. The method is demonstrated by computing the load dependent static transmission error and load sharing of a pair of gears in mesh.

INTRODUCTION

Contact between bodies is a very complex phenomenon. Not only is the very nature of contact and friction not completely understood, but even the very simplest friction models like Coulomb friction can pose difficulties that are only now being overcome.

In the area of contact analysis in the context of finite elements, much progress has been made and previous researchers have used a variety of techniques, including various iterative and mathematical programming methods, the method of Lagrange multipliers and the penalty method.

Penalty methods have been used by Endo *et al.* [1] and Simo *et al.* [2] to enforce contact constraints. Another possibility is the use of Lagrange multipliers to enforce these constraints. Bathe and Chaudhury [3, 4] applied the method of Lagrange multipliers to solve the two- and three-dimensional dynamic frictional contact problem. In a recent paper, Chen and Tsai [5] used a mathematical programming technique along with Lagrange multipliers to enforce sliding contact conditions. Stein *et al.* [6] have used a perturbed Lagrangian technique to enforce contact conditions in the postbuckling analysis of shells. Glowinski *et al.* [7] showed the use of an 'augmented' Lagrangian formulation of the penalty method.

Nour-Omid and Wriggers [8] demonstrated the use of a two-level iteration scheme. They described existing methods such as the method of Lagrange multipliers and the penalty method and demonstrated how a two-level scheme can be used to efficiently solve the equations that result. Numerical examples were given for two-dimensional fractionless contact between two bodies. Rahman *et al.* [9], Torstenfelt [10, 11] and Zolti [12] also showed the use of techniques that iteratively determine the contact conditions.

Gap elements such as those of Mazurkiewicz and Ostachowicz [13], the contact elements by Mehlhorn *et al.* [14] and the contact elements of Wunderlich *et al.* [15] allow the analysis of contact within a generalized finite element code such as ADINA.

Mathematical programming methods such as quadratic programming have long been used in structural analysis for plastic limit analysis (de Freitas [16]). Chand *et al.* [17] and Lee and Kwak [18] use a modified Simplex quadratic programming method and Hung and de Saxce [19] applied a mathematical programming technique to the frictionless static contact problem, and Talaslidis and Panagiotopaulos [20] have developed another mathematical programming technique which uses the associated variational inequality to solve the dynamic frictional contact problem. Stavroulakis *et al.* [21] have expressed the problem of static frictional contact between a pipeline and a rigid sea-floor as a quadratic programming problem and have suggested an iterative method for its solution.

For mechanisms which, due to kinematic constraints, move close to predetermined trajectories the dynamic contact problem can be *linearized* to a series of linear programming problems, for which efficient solution techniques exist. Gears, cams, shafts and many types of linkages would fall into this category. This technique of converting the contact problem into a linear programming problem has hitherto not been applied to problems with friction, and this paper shows that it is a natural and elegant way of dealing with frictional contact and should make tangible the solution of large contact problems.

LINEARIZATION OF THE CONTACT CONDITIONS

The working assumption for the following development is that the bodies undergo deformations that are small in comparison to the dimensions of the body, and that they move along trajectories that are very close to ideal motion (such as in the work by Agrawal and Shabana [22]). The deviation from ideal motion does not substantially affect the direction of the normal vector or the direction of the relative sliding velocity vector of the bodies at the candidate points for contact. Friction is assumed to obey Coulomb's law.

Let there be a total of N bodies in contact with each other. Consider body i which has been discretized into a finite element mesh. It is assigned a reference frame $\mathbf{X}_i$ to which it is attached by means of appropriate constraints. Let $\mathbf{r}_i$ be the vector of unconstrained degrees of freedom of the finite element mesh with respect to $\mathbf{X}_i$, and let $\boldsymbol{\phi}_i$ be the corresponding load vector. Then the stiffness equation relating $\boldsymbol{\phi}_i$ and $\mathbf{r}_i$ is of the form:

$$[K_i]\mathbf{r}_i = \boldsymbol{\phi}_i.$$

The stiffness matrix $[K_i]$ can be obtained by conventional finite element methods. If the body has been properly attached to its reference frame, then this stiffness matrix will be invertible.

Each reference frame $\mathbf{X}_i$ has three degrees of freedom, of which any of the degrees may be constrained. Let $\boldsymbol{\theta}_i$ be a vector containing the unconstrained degrees of freedom of $\mathbf{X}_i$ and let $\boldsymbol{\lambda}_i$ be a vector containing the corresponding generalized loads. Vectors $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ are defined as

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T, \ldots, \boldsymbol{\theta}_N^T)^T \quad \text{and} \quad \boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^T, \boldsymbol{\lambda}_2^T, \ldots, \boldsymbol{\lambda}_N^T)^T.$$

Let L be the number of elements in each of these vectors. Let $(P_{1,m}, P_{2,m})$ be a pair of points on bodies $i_{1,m}$ and $i_{2,m}$, respectively, where $m = 1, 2, \ldots M$. Here M is the total number of such pairs. These pairs of points are candidates for contact and will be termed candidate point pairs (CPP). A common surface normal to the two bodies passes through points $P_{1,m}$ and $P_{2,m}$, as shown in Fig. 1. Let ϵ_m be the initial separation of the two bodies at the CPP m along the common normal before elastic deformation, assuming that the bodies with their reference frames are at their kinematically computed positions, and let δ_m be the increase in separation of the bodies due to elastic deformation along the common normals. Let d_m be the final separation along the common normal and p_m be the compressive normal force at the CPP m. Let f_m be the magnitude of the frictional force in the direction of the relative velocities. (It is always a positive scalar.)

Hence,

$$d_m = \epsilon_m + \delta_m \geqslant 0 \quad \text{and} \quad p_m \geqslant 0.$$

Assuming that sliding contact takes place,

$$f_m = \mu p_m$$

and either

$$p_m = 0 \quad \text{or} \quad d_m = 0.$$

For conciseness, let

$$\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_M)^T, \quad \boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_M)^T,$$

$$\mathbf{d} = (d_1, d_2, \ldots, d_M)^T, \quad \mathbf{p} = (p_1, p_2, \ldots, p_M)^T$$

$$\mathbf{f} = (\mathrm{f}_1, \mathrm{f}_2, \ldots, \mathrm{f}_M)^{\mathrm{T}}.$$

From the geometry of the system, it is possible to use the stiffness matrices of the bodies with respect to their reference frames and the relative orientations of these reference frames to arrive at a linear relationship that shows the dependence of the vector $\boldsymbol{\delta}$ on $\mathbf{p}$, $\mathbf{f}$ and $\boldsymbol{\theta}$ for small deformations:

$$\boldsymbol{\delta} = [A_p]\mathbf{p} + [A_f]\mathbf{f} + [C]\boldsymbol{\theta}.$$

Likewise, from equilibrium considerations, it is possible to obtain a relationship between the generalized loads on the reference frame and the contact forces. This relationship will also be linear.

$$\boldsymbol{\lambda} = [B_p]\mathbf{p} + [B_f]\mathbf{f}.$$

Hence

$$\begin{aligned}\boldsymbol{\delta} &= ([A_p] + \mu[A_f])\mathbf{p} + [C]\boldsymbol{\theta} \\ &= [A]\mathbf{p} + [C]\boldsymbol{\theta} \text{ (say)}\end{aligned}$$

and

$$\boldsymbol{\lambda} = ([B_p] + \mu[B_f])\mathbf{p} = [B]\mathbf{p} \text{ (say)}.$$

The final separation after deformation is, therefore,

$$\mathbf{d} = \boldsymbol{\epsilon} + [A]\mathbf{p} + [C]\boldsymbol{\theta}.$$

The contact problem may then be stated as:

Solve

$$\boldsymbol{\lambda} = [B]\mathbf{p} \quad \text{and} \quad \boldsymbol{\epsilon} = \mathbf{d} - [A]\mathbf{p} - [C]\boldsymbol{\theta}$$

for $\mathbf{d}$, $\mathbf{p}$ and $\boldsymbol{\theta}$

given the separations $\boldsymbol{\epsilon}$ and loads $\boldsymbol{\lambda}$

subject to the conditions $\mathbf{p} \geqslant 0$, $\mathbf{d} \geqslant 0$ and

for each CPP $m = 1, 2, 3, \ldots, M$,

either $d_m = 0$ or $p_m = 0$. (1)

SOLUTION OF THE CONTACT EQUATIONS

Technique 1

A straightforward way of solving the contact equations [eqns (1)] would be to follow the method of Conry and Seireg [23, 24] and pose it in the standard form for linear programming problems [25], which can then be solved by a modified version of the Simplex method. Introduce the new variables $\boldsymbol{\theta}^+$ and

$\boldsymbol{\theta}^-$ such that

$$\boldsymbol{\theta} = \boldsymbol{\theta}^+ - \boldsymbol{\theta}^-$$

and the 'artificial' variables $\mathbf{z}$ (see Hadley [25]). Physically, the artificial variables $\mathbf{z}$ represent the equilibrium imbalance. Then the contact problem in the standard linear programming form would be:

$$\text{Minimize} \sum_{i=1}^{L} z_i,$$

subject to

$$\begin{bmatrix} [I] & 0 & [B] & 0 & 0 \\ 0 & [I] & -[A] & -[C] & [C] \end{bmatrix} \begin{Bmatrix} \mathbf{z} \\ \mathbf{d} \\ \mathbf{p} \\ \boldsymbol{\theta}^+ \\ \boldsymbol{\theta}^- \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\epsilon} \end{Bmatrix}$$

$$\mathbf{z} \geqslant 0, \quad \mathbf{d} \geqslant 0, \quad \mathbf{p} \geqslant 0, \quad \boldsymbol{\theta}^+ \geqslant 0, \quad \boldsymbol{\theta}^- \geqslant 0$$

and the additional condition:

either

$$d_m = 0 \quad \text{or} \quad p_m = 0$$

for each

$$\text{CPP } m = 1, 2, 3, \ldots, M. \tag{2}$$

The additional condition is imposed in the Simplex procedure by requiring that no vector be allowed to move into the basis if it causes a violation of this condition. However, due to this modification of the Simplex algorithm, there is no longer a guarantee that the procedure will always find the solution when there is one. It was observed for several cases that the procedure reached a deadlock even though a solution existed, i.e. the condition imposed on the movement of vectors prevents the procedure from moving towards the solution.

Technique 2

This technique is also like the Simplex method, but with major changes. While the Simplex technique moves from one basic feasible solution to another while eliminating the artificial variable z_i from the basis, this technique will eliminate these artificial variables in its first stage, ending up with values for the other variables which might violate the non-negativity conditions. In its second stage, an iterative procedure 'connects' and 'disconnects' the bodies at the appropriate CPPs until all non-negativity constraints are satisfied.

First an initial 'tableau' $[T]$ (see [25]) is set up:

$$[T] = \begin{bmatrix} [I] & 0 & [B] & 0 \\ 0 & [I] & -[A] & -[C] \end{bmatrix}.$$

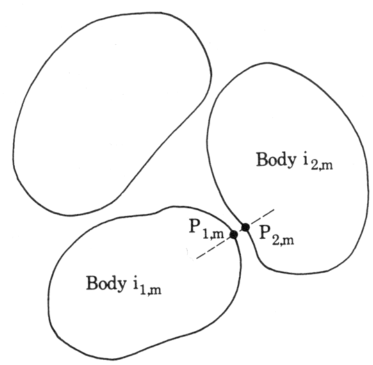

Fig. 1. The candidate point pair $(P_{1,m}, P_{2,m})$.

The starting basic solution vector $\mathbf{s}$ is

$$\mathbf{s} = \begin{Bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\epsilon} \end{Bmatrix}$$

corresponding to a starting basis

$$\mathbf{b} = \begin{Bmatrix} 1 \\ 2 \\ \vdots \\ M+L \end{Bmatrix},$$

in other words, the starting solution is:

$$\mathbf{x} = \begin{Bmatrix} \mathbf{z} \\ \mathbf{d} \\ \mathbf{p} \\ \boldsymbol{\theta} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\epsilon} \\ 0 \\ 0 \end{Bmatrix}.$$

If $T_{l,k} \neq 0$ then it is possible to move from one basic solution and its tableau to the new basic solution and tableau that would be obtained by eliminating the variable x_{b_l} from the basis and replacing it by the variable x_k as follows (apostrophes denote new values):

$$s'_i = s_i - \frac{T_{i,k}}{T_{l,k}} s_l$$

$$i = 1, 2, 3, \ldots, (M+L) \quad \text{and} \quad i \neq l,$$

$$s'_l = s_l / T_{l,k},$$

$$T'_{i,j} = T_{i,j} - \frac{T_{i,k}}{T_{l,k}} T_{l,j}$$

$$i = 1, 2, 3, \ldots, (M+L) \quad \text{and} \quad i \neq l,$$

$$j = 1, 2, 3, \ldots, 2(M+L)$$

$$T'_{l,j} = T_{l,j} / T_{l,k}$$

$$j = 1, 2, 3, \ldots, 2(M+L)$$

Table 1. An algorithmic description of Stage I

```
procedure stage I
for i = 1 to L do
    for some j, such that the pivot ≠0 and d_j is in the basis,
    do
        replace z_i from the basis by p_j.
        replace d_j from the basis by θ_i.
    enddo
enddo
```

$$b_i' = b_i$$

$$i = 1, 2, 3, \ldots, (M+L) \quad \text{and} \quad i \neq l$$

$$b_l' = k$$

and

$$x_i' = s_j \quad \text{if there exists a } j \text{ such that}$$

$$b_j = i, j = 1, 2, 3, \ldots, M+L,$$

$$x_i' = 0 \text{ otherwise.}$$

Unless other checks are made, this transformation *does not* ensure that the basic variables are non-negative. The term $T_{l,k}$ will be called the pivot for this transformation.

Stage I: Initially, the variables that are in the basis (and are therefore non-zero) are **z** and **d**. At the end of Stage I, **z** will be eliminated from the basis and will hence be forced to zero. In other words, the system of bodies will be in equilibrium. Table 1 shows an algorithmic description of this stage.

Stage II: At the beginning of Stage II, the system is in equilibrium, although some of the non-negativity constraints may not be satisfied, meaning that there may be CPPs which carry a negative compressive load or have a negative separation. Also, all of the variables θ_i $i = 1, 2, \ldots, L$ are in the basis along with one of d_m or p_m, for each $m = 1, 2, 3, \ldots, M$. If d_m is in the basis (and is therefore possibly non-zero), then this means that the CPP m is 'disconnected' and carries no load and that the separation between the two bodies at this CPP in the surface normal direction is d_m. If p_m is in the basis (and is therefore possibly non-zero), then the CPP m is 'connected' and carries the compressive load p_m. But at the end of Stage I the values of the variables may be negative. At the end of Stage II, all non-negativity constraints are satisfied. Table 2 shows an algorithmic description of Stage II. If the system of contact bodies is unstable (such as the two-body system in Fig. 2) the procedure diagnoses this condition. Such an instability may also be caused by improperly chosen CPPs. When the procedure terminates, the solution is available in the vector **x**.

NUMERICAL EXAMPLE

The case of gears with involute profiles is a particularly good example of the application of this method. Let body 1 be the 'input' gear and body 2 be the 'output' gear. Their reference frames are attached to rigid shafts that rotate with the gears. Reference frame $\mathbf{X}_1$ has no unconstrained degrees of freedom because the location of its origin as well as its angular orientation are *prescribed.* Reference frame $\mathbf{X}_2$ has its origin at a prescribed location, but is allowed to rotate by a small amount about its kinematically computed angular position. Thus it has one (rotational) degree of freedom θ_0. The generalized load associated with this degree of freedom is the output torque M_0. θ_0 is defined to be the transmission error of the gear pair and will depend on the prescribed value for the output torque M_0.

As mentioned earlier, it is straightforward to compute the matrices $[A]$ and $[C]$ such that

$$\mathbf{d} = \boldsymbol{\epsilon} + [A]\mathbf{p} + [C]\theta_0.$$

Taking moments about the output shaft axis yields an equilibrium equation of the form

$$M_0 = [B]\mathbf{p}.$$

Figures 3 and 4 show the finite element model of the two-gear system. Both gears have 20 teeth, a pressure angle of 20 degrees, an addendum constant of 0.75, dedendum constant 1.4 and a diametral pitch of 10 in^{-1}. The center distance is 2.0 in. Figure 5 shows a sample choice of candidate point pairs for a particular orientation of the gears. The maximum allowable distance between the two points in the pairs has been exaggerated for this figure. Such pairing has to be carried out for each individual orientation of the gears.

The output gear was loaded with a known moment M_0 and the contact equations were solved for 50 different orientations of the gear within each tooth cycle. The input torque M_i and the transmission error θ_0 were thus obtained. Figure 6 shows two different curves, one for when the coefficient of friction was chosen to be zero and the other for when it was 0.3. This coefficient of friction is much higher than would be observed for lubricated gears, but was chosen to illustrate the effect of friction. In both cases, the

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    enddo
enddo
```

$$b_i' = b_i$$

$$i = 1, 2, 3, \ldots, (M + L) \quad \text{and} \quad i \neq l$$

$$b_l' = k$$

and

$x_i' = s_j$ if there exists a j such that

$$b_j = i, j = 1, 2, 3, \ldots, M + L,$$

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Stage II: At the beginning of Stage II, the system is in equilibrium, although some of the non-negativity constraints may not be satisfied, meaning that there may be CPPs which carry a negative compressive load or have a negative separation. Also, all of the variables θ_i $i = 1, 2, \ldots, L$ are in the basis along with one of d_m or p_m, for each $m = 1, 2, 3, \ldots, M$. If d_m is in the basis (and is therefore possibly non-zero), then this means that the CPP m is 'disconnected' and carries no load and that the separation between the two bodies at this CPP in the surface normal direction is d_m. If p_m is in the basis (and is therefore possibly non-zero), then the CPP m is 'connected' and carries the compressive load p_m. But at the end of Stage I the values of the variables may be negative. At the end of Stage II, all non-negativity constraints are satisfied. Table 2 shows an algorithmic description of Stage II. If the system of contact bodies is unstable (such as the two-body system in Fig. 2) the procedure diagnoses this condition. Such an instability may also be caused by improperly chosen CPPs. When the procedure terminates, the solution is available in the vector **x**.

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Table 2. An algorithmic description of Stage II

```
Procedure stage II
repeat until (p_i ≥ 0 and d_i ≥ 0 for all i = 1, 2, . . . , M)
  if (for some i, d_i < 0) then
    connect (i)
  endif
  if (for some i, p_i < 0) then
    if (appropriate pivot ≠ 0) then
      disconnect (i)
    else
          Comment
          If the pivot required to disconnect CPP i is zero, it means
          that by disconnecting CPP i, the system becomes statically
          unstable. Another (disconnected) CPP j should be found,
          which when connected, will have a positive p_j and will allow
          CPP i to be disconnected. If no such CPP j exists, then the
          system of bodies is either inherently statically unstable, or
          the CPPs have not been chosen properly.
          Endcomment
          for (all j such that d_j is in the basis) repeat
            connect (j)
            if (required pivot is still = 0) then
              disconnect (j)
              Goto Next
            endif
            disconnect (i)
            if (p_i ≥ 0) then
              Goto Found
            else
              connect (i)
              disconnect (j)
            endif
Next::      continue
          endfor
          Write Message "Statically unstable system"
          return
Found::   disconnect (i)
      endif
    endif
endrepeat
endprocedure

Procedure connect (i)
replace d_i from the basis by p_i.
endprocedure

Procedure disconnect (i)
replace p_i from the basis by d_i.
endprocedure
```

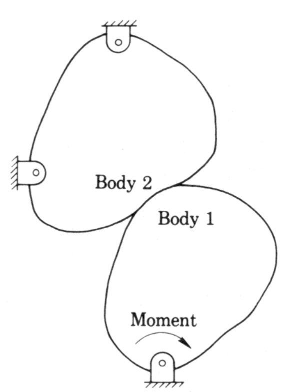

Fig. 2. Two bodies in statically unstable contact.

output torque is 1000 lb in. Figure 7 shows transmission error curves for $M_0 = 1000$ lb in. with the coefficient of friction $= 0.0$ and 0.3, and for $M_0 = 100$ lb in. with the coefficient of friction $= 0.3$. The transmission error curves clearly show parts where single-tooth contact and double-tooth contact takes place. As expected, the case with zero coefficient of friction gives transmission error curves that are symmetric. When a coefficient of friction is included, the curves become skewed. Also as expected, at the point where the direction of the frictional force changes, a jump in the transmission error occurs and the input torque M_i equals the output torque M_0 exactly.

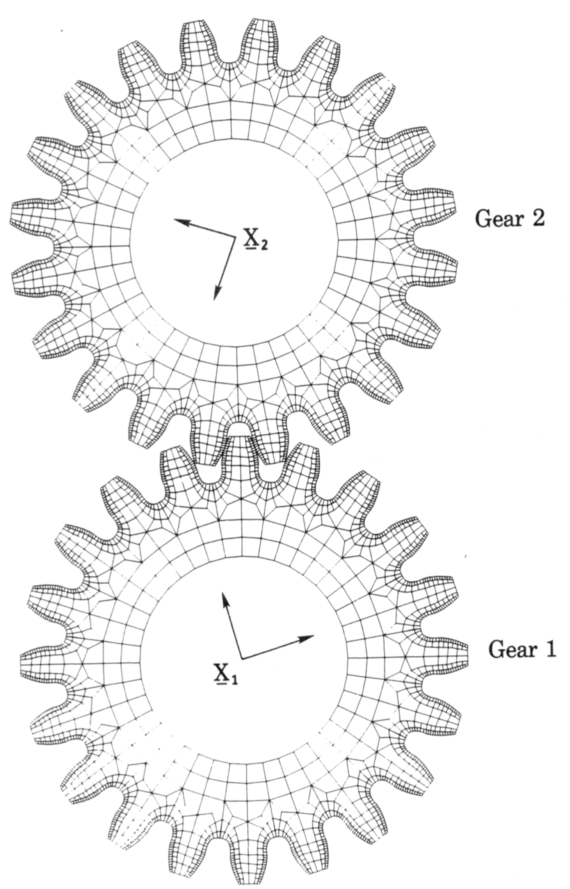

Fig. 3. Finite element model of a two-gear system in contact.

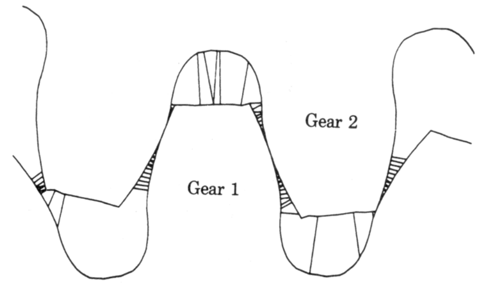

Fig. 5. Candidate point pairs chosen for a particular position of the contacting gears.

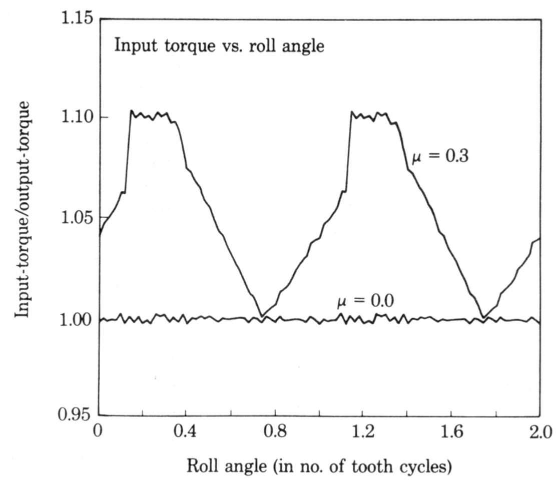

Fig. 6. Dependence of input torque on the coefficient of friction.

The transmission error profile is similar in character to experimentally observed curves. Slight 'noise' has, however, been introduced in the transmission error and input torque curves due to the fact that the gear-tooth profiles have been approximated by straight lines. It is encouraging note that this 'noise' in the transmission error is much smaller than the transmission error itself. In order to model the involute profile as closely as possible, a special five-noded linear transition element is used. This allows a maximum possible number of nodes on the surface while keeping the total number of nodes as low as possible. Figure 8 shows the total of the normal surface loads on the CPPs of each individual

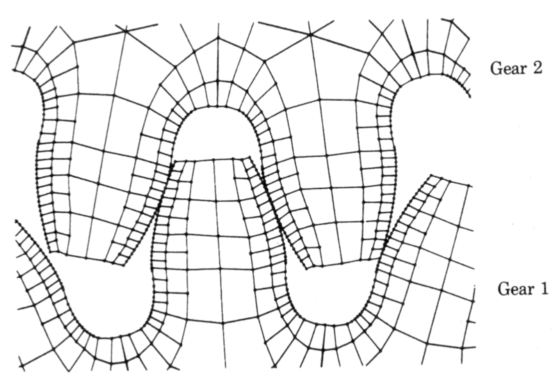

Fig. 4. Finite element model of a two-gear system in contact.

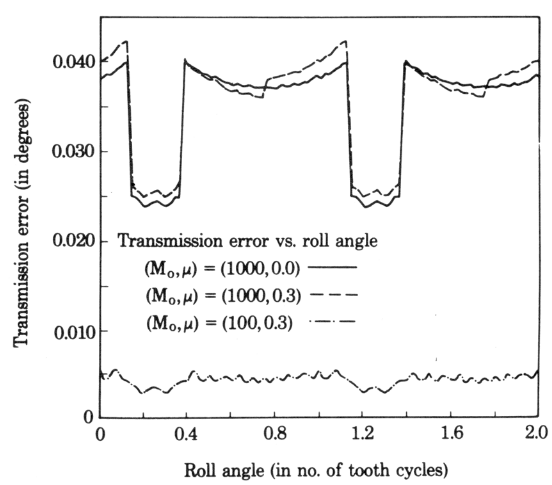

Fig. 7. Dependence of transmission error on the load torque.

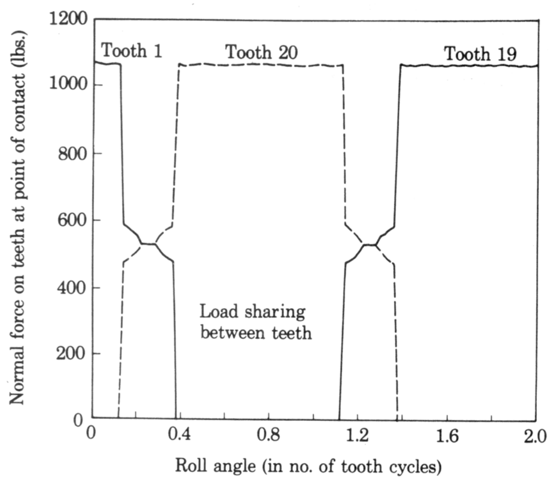

Fig. 8. Load sharing between gear teeth.

gear tooth as the gears roll through, for the case with $M_0 = 1000$ lb in. and coefficient of friction $= 0.0$. As expected, these coincide exactly with the theoretical value of 1064 lb.

CONCLUSIONS

It is the authors' opinion that the Simplex type algorithm described here is a natural choice for contact problems. Although the discussion was applicable only to situations in which the bodies were 'kinematically constrained', a similar algorithm can be used inside an outer iterative loop for more general contact procedures. The stiffness matrices do not need to be decomposed for each time step, as in the penalty method or in the method of Lagrange multipliers. It is also felt that this and similar techniques will help make larger contact problems in three-dimensions with a large number of candidate point pairs easier to solve. Although the scope of this paper necessitated considering only sliding friction, it is possible to treat contact with sticking friction in a similar manner.

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